Prerequisites

Pollack&Stump, Chapter 12, or Zangwill 22

28 Rindler Horizon

A Reality Escape Pod (R.E.P.) is a fully self-sustained capsule with a drive delivering constant acceleration a, as felt by the pilot, without ever running out of fuel.

(a) The R.E.P. starts its journey at rest at the origin of the world's rest frame $\mathcal{F}$. Find the world line $x^\mu(\tau)$ in the frame $\mathcal{F}$ as function of the pilot's proper time τ.

(b) Show that there is a time T in the frame $\mathcal{F}$ after which no message can reach the R.E.P. What would be the required acceleration a in multiples of earth's gravity for the pilot to escape any news or consequences from an event scheduled on earth 58 days after departure? These events are said to be hidden by the *Rindler Horizon*.

(c) Despite the notable acceleration forces, the pilot lives for another 25 years. Afterwards, the R.E.P. self-destructs. How long after the departure will the luminous explosion be seen on earth?

29 Star distribution in relativistic flight

After delivering a present to an earth citizen in a Reality Escape Pod, Santa Claus is in quite a hurry and has to travel at relativistic speed v back towards earth. Help his rocket-reindeer Rudi to find his way home among the stars, distorted by relativistic effects.

(a) Let θ and θ' be a star's apparent angle relative to the direction of flight at rest and in motion, respectively. Show that

$$\frac{\sin\theta'}{\cos\theta'} = \frac{\sin\theta}{\gamma(\cos\theta + \beta)}.$$

(b) Let ω and ω' be the observed frequency of light from a star at rest and in motion, respectively. Assuming all stars are at rest, find the Doppler shift ω/ω' as a function of θ.

30 Cosmological redshift

Assume a linearly expanding universe whose objects are all in the same inertial frame. Their coordinates are thus given by

$$x^\mu = (ct, a(t)\,\boldsymbol{R}),$$

where $a(t) = H_0 t$ is the scaling function of space itself, depending on the cosmic time t, and where $\boldsymbol{R}$ is today's position of an object. The constant H_0 is called *Hubble's constant.*

(a) Objects at distances R beyond c/H_0 recede faster than the speed of light, purely due to the expansion of space. Show that a light signal emitted today from such a distant object will still be observable on earth at $\boldsymbol{R} = \boldsymbol{0}$. Hint: Instantaneously, light always propagates with the speed c.

(b) Let ω be the frequency of light emitted in part (a). Derive the frequency of the observed light ω' and give the ratio ω/ω' as a function of today's speed of recession $v_\mathrm{r} = RH_0$ of the emitting object. Hint: The speed of the expansion of space can be ignored for the local propagation of light.

(c) Under these assumptions, what was the radius of the hydrogen recombination surface that emitted the blackbody radiation at roughly $T_\mathrm{e} = 3000\,\mathrm{K}$ which can now be observed as the cosmic microwave background at $T_\mathrm{o} = 3\,\mathrm{K}$? What is its radius today?

$H_0^{-1} \approx 13 \times 10^9$ a.

31 Covariant charge and current density

Show by explicit calculation that the formulae for the charge and current densities of a collection of point charges,

$$\rho(t, \boldsymbol{r}) = \sum_{k=1}^{N} q_k \, \delta(\boldsymbol{r} - \boldsymbol{r}_k(t)) \qquad \boldsymbol{j}(t, \boldsymbol{r}) = \sum_{k=1}^{N} q_k \boldsymbol{v}_k(t) \, \delta(\boldsymbol{r} - \boldsymbol{r}_k(t))$$

have exactly the same form when we boost from frame $\mathcal{F}$ to frame $\mathcal{F}'$ by a velocity v_0. Hint: Show first that $\delta[\boldsymbol{r} - \boldsymbol{r}'_k(t')] = \gamma(1 - \boldsymbol{v}_0 \cdot \boldsymbol{v}_k/c^2)\, \delta[\boldsymbol{r} - \boldsymbol{r}_k(t)]$ by evaluating the Jacobian determinant in the volume element transformation from $\mathrm{d}^3 R$ to $\mathrm{d}^3 r'$, where $\boldsymbol{R} = \boldsymbol{r} - \boldsymbol{r}_k(t)$.

32 Stress–energy tensor

The stress–energy tensor of the electromagnetic field $F^{\mu\nu}$ is given by

$$T^{\mu\nu} = \frac{1}{\mu_0} \left(F^{\mu\rho} F^{\nu}{}_{\rho} - \frac{1}{4} \eta^{\mu\nu} F^{\rho\sigma} F_{\rho\sigma} \right)$$

with the Minkowski metric tensor η having the signature $(-, +, +, +)$. Evaluate the Lorentz invariant $T^{\mu\nu} T_{\mu\nu}$ and identify a type of electromagnetic field where this invariant is zero.

33 Electromagnetic field tensor due to a moving charge

A charge q moves with constant velocity $\boldsymbol{v} = v\hat{\boldsymbol{x}}$. Define the antisymmetric tensor $r^{\mu\nu}$ by

$$r^{\mu\nu} = \frac{1}{c}(u^\mu x^\nu - u^\nu x^\mu)$$

where u^μ is the 4-velocity of q. Also define $r^2 = -r^{\mu\nu} r_{\mu\nu}/2$.

(a) Prove that

$$r^2 = \frac{(x - vt)^2}{1 - v^2/c^2} + y^2 + z^2.$$

(b) Show that the electromagnetic field tensor of the charge has the elegant form

$$F^{\mu\nu} = \frac{q}{4\pi\epsilon_0 c} \frac{r^{\mu\nu}}{r^3}.$$

Merry Christmas and a Healthy New Year!