Vienna University of Technology SS 2019
Institute for Analysis and Scientific Computing
Dr. B. Nectoux, Prof. A. Arnold

Exercise Sheet 2 "Nonlinear Partial Differential Equations"
(nonlinear elliptic PDE, comparison principle, Sobolev-functions)

Exercise 1. Consider the boundary-value-problem (BVP)

$$L(u) = f \text{ in } I\,, \quad u(\alpha) = c\,, \quad u(\beta) = d\,, \tag{1}$$

for some function $f \in C^1(I)$, an interval $I = (\alpha, \beta) \subset \mathbb{R}$, constants c, $d \in \mathbb{R}$, and a quasi-linear elliptic (w.r.t u) differential operator $L(u) = a(x, u_x)u_{xx} + b(x, u, u_x)$. If the functions a, b are sufficiently smooth and the function b is monotone decreasing with respect to u then classical solutions of (1) are unique, see Corollary 2.2 and Theorem 2.1 in the lecture notes.
Find an example of a BVP (1) with some function b, which is not monotone decreasing in u, such that BVP (1) does not have a unique classical solution.

Exercise 2. Consider a positive constant $R > 0$ and a bounded domain $\Omega \subset \mathbb{R}^n$ with smooth boundary. Show that there exists a unique *positive* classical solution of the BVP

$$-\Delta u = R^2 - u^2 \text{ in } \Omega\,, \qquad u = R \text{ on } \partial\Omega.$$

Moroever, prove that any weak solution u of this BVP satisfies

$$u \leq R \text{ almost everywhere on } \Omega.$$

Exercise 3. Let X, Y, Z be Banach spaces with continuous embeddings $X \hookrightarrow Y \hookrightarrow Z$. Assume there exist $C > 0$ and $0 < \theta < 1$, s.t. for all $u \in X$:

$$\|u\|_Y \leq C\|u\|_X^{1-\theta}\|u\|_Z^{\theta}.$$

Show that

(i) if $(u_k) \subset X$ is bounded with $\lim_{k\to\infty} u_k = u$ in Z then $\lim_{k\to\infty} u_k = u$ in Y.

(ii) If $X \hookrightarrow Z$ is compact, then also $X \hookrightarrow Y$ is compact.

Exercise 4. Suppose $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary and $u \in H^1(\Omega)$. Show that $u^+ \in H^1(\Omega)$ and

$$\nabla(u^+) \;=\; \begin{cases} \nabla u & \text{a.e. on } \{u > 0\}\,, \\ 0 & \text{a.e. on } \{u \leq 0\}\,. \end{cases}$$

Hint: Define $F_\epsilon : \mathbb{R} \to \mathbb{R}$ by:

$$F_\epsilon(z) \;=\; \begin{cases} \sqrt{z^2+\epsilon^2} - \epsilon & \text{for } \; z \geq 0\,, \\ 0 & \text{for } \; z < 0\,. \end{cases}$$

and show that in $D'(\Omega)$, it holds:

$$\text{when } \epsilon \to 0,\; F_\epsilon(u) \to u^+ \text{ and } F'_\epsilon(u)\nabla u \to 1_{\{u>0\}}\nabla u.$$

To conclude, use the following result on the derivative of a composition (see Proposition 9.5 in the book of Haïm Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Springer, 2010):

Lemma 1. *Suppose $\Omega \in \mathbb{R}^n$ is a bounded domain with smooth boundary, $F \in C^1(\mathbb{R})$ with and $F' \in L^\infty(\mathbb{R})$. Let $u \in W^{1,p}(\mathbb{R})$ for $1 \leq p \leq \infty$. Then $F \circ u \in W^{1,p}(\Omega)$ and, in $L^p(\Omega)$, it holds $\nabla(F \circ u) = (F' \circ u)\nabla u$.*

Remark: when Ω is not bounded, this lemma still holds assuming in addition that $F(0) = 0$.

Let us give the proof of the lemma. First notice that $|F \circ u| \leq F(0) + \|F\|_{L^\infty(\mathbb{R})}|u|$. Since Ω is bounded and $u \in L^p(\Omega)$, one deduces that $F \circ u \in L^p(\Omega)$. It remains to show that $\nabla(F \circ u) = (F' \circ u)\nabla u$ holds in $D'(\Omega)$ (the stated results will follow since $(F' \circ u)\nabla u \in L^p(\Omega)$). Let us consider the case when $p < +\infty$ (the case $p = +\infty$ is treated similarly). Let $(u_n)_{n\geq 1} \in C_c^\infty(\mathbb{R}^n)^{\mathbb{N}}$ such that u_n converges to u in $L^p(\Omega)$, and for all compact subset K of Ω, ∇u_n converges to ∇u in $L^p(K)^n$. Let K be a compact subset of Ω and $\phi \in C_c^\infty(\Omega)$ such that ϕ is supported in K. Then, for all $n \geq 1$, by integration by parts, it holds:

$$\int_K F \circ u_n \,\nabla\phi = \int_K (F' \circ u_n)\nabla u_n \;\phi.$$

Up to the extraction of a subsequence, u_n converges almost surely to u in K. Thus, by the dominated convergence theorem, when $n \to \infty$, one has $\int_K F \circ u_n \,\nabla\phi \to \int_K F \circ u \,\nabla\phi$ and $\int_K (F' \circ u_n)\nabla u_n \,\phi \to \int_K (F' \circ u)\nabla u \,\phi$. Consequently, $\int_K F \circ u \,\nabla\phi = \int_K (F' \circ u)\nabla u \,\phi \Rightarrow \nabla(F \circ u) = (F' \circ u)\nabla u$ in $D'(\Omega)$.

Solutions will be discussed on Thursday 21th of March 2019.