27. November 2008
105.057 Finanzmathematik 2: zeitstetige Modelle, Schachermayer

Dauer 90 Minuten, alle Unterlagen sind erlaubt

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1. Determine the price of a digital option with strike K and payoff (5 Pkt.)

$$\mathbf{1}_{\{S_T \leq K\}}$$

at maturity T in the Black-Scholes model. How many units of stock respectively bond are in a replicating portfolio at time $0 \leq t \leq T$?

2. Let W_t be standard Brownian motion. Use Itô's formula to show that the following processes are martingales. What is the name of the second one? (5 Pkt.)

   (a) $X_t = \mathrm{e}^{t/2} \cos W_t$

   (b) $X_t = \mathrm{e}^{\sigma W_t - \sigma^2 t/2}, \qquad \sigma > 0.$

3. Suppose that μ and $\sigma > 0$ are real numbers. Let S be geometric Brownian motion: (5 Pkt.)

$$S_t = S_0 \exp\left(\sigma W_t + (\mu - \tfrac{1}{2}\sigma^2)t\right), \qquad 0 \leq t \leq T.$$

Show that

$$\sum_{j=0}^{m-1} \left(\log \frac{S_{t_{j+1}}}{S_{t_j}}\right)^2$$

converges to $\sigma^2 T$ in probability, if we consider partitions

$$0 = t_0 < t_1 < \cdots < t_m = T$$

of the interval $[0, T]$ whose mesh size

$$\max_{0 \leq j < m} (t_{j+1} - t_j)$$

tends to zero. Why is this result useful for pricing options in practice?

(**Hints:** (1) Recall the quadratic variation of Brownian motion. (2) Use that

$$\max_{0 \leq j < m} |W_{t_{j+1}} - W_{t_j}| \to 0,$$

by the continuity of the paths of Brownian motion.)