1.2 Exercises 2: Date 19.4.2013

1. Suppose the random variables Y_i are independently and identically distributed $N(0, \sigma^2)$ for $i = 1, 2, ..., n$.

   (a) Show that $E(Y_i^2/\sigma^2) = 1$

   (b) Show that $W = (1/\sigma^2) \sum_{i=1}^{n} Y_i^2$ is distributed χ_n^2.

   (c) Show that $E(W) = n$.

   (d) Show that $V = Y_1 / \sqrt{\frac{\sum_{i=2}^{n} Y_i^2}{n-1}}$ is distributed t_{n-1}.

2. Show that the least squares residuals sum to zero: $\sum e_t = 0$ with $e_t = y_t - \hat{\beta} x_t = -(\hat{\beta} - \beta) x_t + (u_t - \bar{u})$ where $y_t = Y_t - \bar{Y}$, $x_t = X_t - \bar{X}$.
   Show $\sum e_t^2 = (\hat{\beta} - \beta)^2 \sum x_t^2 + \sum (u_t - \bar{u})^2 - 2(\hat{\beta} - \beta) \sum x_t (u_t - \bar{u})$ and

   $$E(\sum e_t^2) = \sigma^2{}_u (T - 2) \tag{4}$$

   where $\sigma^2{}_u = E(u_t^2)$. Derive the unbiased estimator of $\sigma^2{}_u$.

3. Use the file **coffee.xls**, which contains average annual coffee sales (y) and prices (x) for 11 years. Produce a cross plot and form an opinion about the relation of the coffee price and the sold quantity.
   Consider the following linear model:

   $$y_i = \alpha + \beta x_i + u_i \tag{5}$$

   for the years $i = 1970, ...1980$. How do you interpret the parameters α and β? Estimate the parameters α and β by the method of least squares using a program of your choice and compute the estimated errors (residuals) $\hat{u}_i$ (or e_i). Draw the estimated regression line into the cross plot. Do the errors look systematic or not?

4. Compute the estimated variance of the residuals $s^2{}_u$ and those of $\hat{\alpha}$ and $\hat{\beta}$ as well as their covariance. How much of the variance of y is explained by the model?

5. Estimate the model (5) without constant term, i.e. $y_i = \beta x_i + u_i$ and $\alpha = 0$. Draw the estimated regression line into the cross plot. What do you find?

6. Estimate the model (5) with transformed data. Deduct the mean of each variable from the data and use these deviations from the mean as new variables. What do you find?

7. Reformulate the model (5) in matrix notation. Define matrix X and derive matrix $X'X$ and its inverse $(X'X)^{-1}$. Compute the vector of least squares parameters $\theta = (\alpha, \beta)'$, $\hat{\theta} = (X'X)^{-1} X' y$.

8. Compute $\sigma^2{}_u (X'X)^{-1}$. How do you interpret the elements of this matrix? Compare with your results for question 4.

9. Read the attached article on the "beta" of a stock und use the data in the file **capm_usa.xls** to estimate the beta for **one** firm of your choice. Estimate the parameters of the following equation:

   $$R - R_f = \alpha + \beta (R_m - R_f) + u$$

The "Beta" of a Stock

A fundamental idea of modern finance is that an investor needs a financial incentive to take a risk. Said differently, the expected return[1] on a risky investment, R, must exceed the return on a safe, or risk-free, investment, R_f. Thus the expected excess return, $R - R_f$, on a risky investment, like owning stock in a company, should be positive.

At first it might seem like the risk of a stock should be measured by its variance. Much of that risk, however, can be reduced by holding other stocks in a "portfolio"—in other words, by diversifying your financial holdings. This means that the right way to measure the risk of a stock is not by its *variance* but rather by its *covariance* with the market.

The capital asset pricing model (CAPM) formalizes this idea. According to the CAPM, the expected excess return on an asset is proportional to the expected excess return on a portfolio of all available assets (the "market portfolio"). That is, the CAPM says that

$$R - R_f = \beta(R_m - R_f), \qquad (4.12)$$

where R_m is the expected return on the market portfolio and β is the coefficient in the population regression of $R - R_f$ on $R_m - R_f$. In practice, the risk-free return is often taken to be the rate of interest on short-term U.S. government debt. According to the CAPM, a stock with a $\beta < 1$ has less risk than the market portfolio and therefore has a lower expected excess return than the market portfolio. In contrast, a stock with a $\beta > 1$ is riskier than the market portfolio and thus commands a higher expected excess return.

The "beta" of a stock has become a workhorse of the investment industry, and you can obtain estimated betas for hundreds of stocks on investment firm Web sites. Those betas typically are estimated by OLS regression of the actual excess return on the stock against the actual excess return on a broad market index.

The table below gives estimated betas for seven U.S. stocks. Low-risk producers of consumer staples like Kellogg have stocks with low betas; riskier stocks have high betas.

Company	Estimated β
Wal-Mart (discount retailer)	0.3
Kellogg (breakfast cereal)	0.5
Waste Management (waste disposal)	0.6
Verizon (telecommunications)	0.6
Microsoft (software)	1.0
Best Buy (electronic equipment retailer)	1.3
Bank of America (bank)	2.4

Source: SmartMoney.com.

[1] The return on an investment is the change in its price plus any payout (dividend) from the investment as a percentage of its initial price. For example, a stock bought on January 1 for $100, which then paid a $2.50 dividend during the year and sold on December 31 for $105, would have a return of $R = [(\$105 - \$100) + \$2.50]/\$100 = 7.5\%$.

Figure 1: from: Stock and Watson (2012) p.160

with R the return of the stock (MOBIL, TEXACO, DEC, PANAM, DELTA,) R_f the risk-free return (RKFREE), and R_m the return of the market portfolio (MARKET). The data are monthly averaged time series from January 1978 until December 1987, 120 observations. First, calculate the risk premia for the firm and the market and then estimate the parameters α and β with a regression program of your choice. Interpret the meaning of your estimates.

10. Test the estimated parameters of exercise 9: Test α against the value of zero and β against the value of one using an error probability of 0.05%. Formulate the null hypothesis and the alternative and interpret the test result.