Nachtest aus Statistischer Physik (Lösung)

22.10.2010, 14:00Uhr

1. (a) Zustandssumme für $N = 1$

$$\begin{aligned} Z_K(m,T,1) &= \mathrm{Sp}\left(e^{-\beta H}\right) \\ &= \exp\left[-\frac{1}{2}\beta J m^2 + \mu\beta m J\right] + \exp\left[-\frac{1}{2}\beta J m^2 - \mu\beta m J\right] \\ &= 2\exp\left[-\frac{1}{2}\beta J m^2\right]\cosh\left[\mu\beta m J\right] \end{aligned}$$

Zustandssumme für N

$$Z_K(m,T,N) = 2^N \exp\left[-\frac{1}{2}\beta N J m^2\right]\cosh^N\left[\mu\beta m J\right]$$

Freie Energie :

$$\begin{aligned} F(m,T,N) &= -k_B T \ln Z_K(m,T,N) \\ &= -N k_B T\left(\ln 2 - \frac{1}{2}\beta J m^2 + \ln\left(\cosh\left[\mu\beta m J\right]\right)\right) \\ &= \frac{1}{2} N J m^2 - N k_B T\left(\ln 2 + \ln\left(\cosh\left[\mu\beta m J\right]\right)\right) \end{aligned}$$

(b) Freie Enthalpie im Nichtgleichgewicht

$$g(m,B,T,N) = \frac{1}{2} N J m^2 - N k_B T\left(\ln 2 + \ln\left(\cosh\left[\mu\beta m J\right]\right)\right) + N m B$$

Zusätzliche Information

Legendre-Transformation :
Gleichgewichtsbedingung

$$\frac{\partial g}{\partial m} = 0 \quad \rightarrow \quad m = m(B,T,N)$$

Freie Enthalpie im Gleichgewicht

$$G(B,T,N) = g(m(B,T,N),B,T,N)$$

Taylor-Entwicklung

$$g \simeq g|_{m=0} + \left.\frac{\partial g}{\partial m}\right|_{m=0} m + \frac{1}{2}\left.\frac{\partial^2 g}{\partial m^2}\right|_{m=0} m^2 + \frac{1}{3!}\left.\frac{\partial^3 g}{\partial m^3}\right|_{m=0} m^3 + \frac{1}{4!}\left.\frac{\partial^4 g}{\partial m^4}\right|_{m=0} m^4 + \cdots$$

$$g(0,B,T,N) = -N k_B T \ln 2$$

$$\frac{dg}{dm} = N J m - N\mu J \tanh\left[\mu\beta m J\right] + N B \rightarrow \frac{dg}{dm}(0,B,T,N) = N B$$

$$\frac{d^2 g}{dm^2} = N J - N\beta(\mu J)^2 \frac{1}{\cosh^2\left[\mu\beta m J\right]} \rightarrow \frac{d^2 g}{dm^2}(0,B,T,N) = N J\left(1 - \beta\mu^2 J\right)$$

$$\frac{d^3g}{dm^3} = 2N\beta^2(\mu J)^3 \frac{\sinh[\mu\beta m J]}{\cosh^3[\mu\beta m J]} \rightarrow \frac{d^3g}{dm^3}(0,B,T,N) = 0$$

$$\frac{d^4g}{dm^4} = 2N\beta^3(\mu J)^4 \frac{1-2\sinh^2[\mu\beta m J]}{\cosh^4[\mu\beta m J]} \rightarrow \frac{d^4g}{dm^4}(0,B,T,N) = 2N\beta^3(\mu J)^4$$

Deshalb ist die freie Enthalpie

$$g(m,B,T,N) \simeq -Nk_BT\ln 2 + \frac{1}{2}NJ\left(1-\beta\mu^2 J\right)m^2 + \frac{1}{12}N\beta^3(\mu J)^4 m^4 + NBm$$

Wenn $B=0$,

$$\frac{\partial g}{\partial m} = NJ\left(1-\beta\mu^2 J\right)m + \frac{1}{3}N\beta^3(\mu J)^4 m^3$$

Weil $(1/12)N\beta^3(\mu J)^4 > 0$, hat die Gleichung $\partial g/\partial m = 0$ eine Lösung $m=0$ für $1 > \beta\mu^2 J$ und drei Lösungen

$$m = 0, \pm\sqrt{\frac{3J(\beta\mu^2 J - 1)}{\beta^3(\mu J)^4}}$$

für $1 < \beta\mu^2 J$. Das System zeigt einen Phasenübergang zweiter Ordnung. Bei der kritischen Temperatur $(1/2)NJ(1-\beta\mu^2 J) = 0$, d.h.

$$T_c = \frac{1}{k_B}\mu^2 J\,.$$

(c) Gleichgewichtsbedingung : $dg/dm = 0$

$$NJ\left(1-\beta\mu^2 J\right)m + \frac{1}{3}N\beta^3(\mu J)^4 m^3 + NB = 0$$

Ableitung der Gleichgewichtsbedingung nach B

$$NJ\left(1-\beta\mu^2 J\right)\frac{dm}{dB} + N\beta^3(\mu J)^4 m^2\frac{dm}{dB} + N = 0$$

Deshalb,

$$\frac{dm}{dB} = -\frac{1}{\beta^3(\mu J)^4 m^2 + J(1-\beta\mu^2 J)}$$

Wenn $T > T_c$,

$$\left.\frac{dm}{dB}\right|_{B=0} = \frac{1}{J(\beta\mu^2 J - 1)}$$

Wenn $T < T_c$,

$$\left.\frac{dm}{dB}\right|_{B=0} = \frac{1}{2J(1-\beta\mu^2 J)}\,.$$

2. (a) Phasenraumvolumen mit $H < E$:

$$\begin{aligned}
\Phi(E) &= \int_{H<E} d^N p_x d^N p_y d^N L d^N x d^N y d^N \theta \\
&= (2\pi V)^N (2m)^{N/2}(2m)^{N/2}(2I)^{N/2}\int_{\sum_{i=1}^{3N} X_i^2 < E} d^{3N}dX \\
&= (4\pi m V)^N (2I)^{N/2}\frac{\pi^{3N/2}E^{3N/2}}{\Gamma(3N/2+1)}
\end{aligned}$$

Anzahl der Zustände

$$\begin{aligned}
\Omega(E) &= \frac{1}{2N!h^{3N}}\Omega_V(E) = \frac{1}{2N!h^{3N}}\Delta\frac{d}{dE}\Phi(E) \\
&= \frac{1}{2N!h^{3N}}\Delta\frac{3}{2}N(4\pi mV)^N(2I)^{N/2}\frac{\pi^{3N/2}E^{3N/2-1}}{\Gamma(3N/2+1)}
\end{aligned}$$

oder, im Limes $N \gg 1$,

$$\Omega(E) \simeq \frac{1}{2N!h^{3N}}\Phi(E) = E^{3N/2}C(N,V)$$

(b) Entropie :

$$S = k_B \ln\Omega = \frac{3}{2}Nk_B \ln E + k_B \ln C(N,V)$$

Temperatur :

$$T = \left(\frac{\partial S}{\partial E}\right)^{-1}_{V,N} = \left(\frac{3Nk_B}{2E}\right)^{-1} = \frac{2E}{3Nk_B}$$

Energie

$$E = \frac{3}{2}Nk_BT$$

Wärmekapazität :

$$C_V = \frac{3}{2}Nk_B$$

3. Wahrscheinlichkeitsdichte :

$$\rho(\vec{r}_1,\vec{p}_1,\cdots,\vec{r}_N,\vec{p}_N;N) = \frac{1}{Z_{\mathrm{GK}}}\frac{1}{N!h^{3N}}e^{-\beta(H-\mu N)}$$

Normierung :

$$\begin{aligned}
Z_{\mathrm{GK}} &= \sum_N \int d^{3N}rd^{3N}p\frac{1}{N!h^{3N}}e^{-\beta(H-\mu N)} \\
&= \sum_N \frac{V^N}{N!h^{3N}}z^N\left(\int dpe^{-\beta p^2/(2m)}\right)^{3N} \\
&= \sum_N \frac{1}{N!}\left(\frac{zV}{h^3}(2\pi mk_BT)^{3/2}\right)^N \\
&= \sum_N \frac{1}{N!}\left(\frac{zV}{\lambda_T^3}\right)^N \\
&= \exp\left(z\frac{V}{\lambda_T^3}\right)
\end{aligned}$$

Deshalb

$$\rho(\vec{r}_1,\vec{p}_1,\cdots,\vec{r}_N,\vec{p}_N;N) = \frac{1}{N!h^{3N}}e^{-\beta(H-\mu N)}\exp\left(-z\frac{V}{\lambda_T^3}\right)$$