

Complexity Theory

VU 181.142, SS 2015

Homework Assignment 1

Name: N.N.
Matr-Nr: xxxxxxxx
Begin: 17 March, 2015
Submission Deadline: 14 April, 2015
send to: complexity@dbai.tuwien.ac.at
Maximum credits: 10

In the lecture “Formale Methoden der Informatik”, the following Cook reduction from **co-2-SAT** to the **REACHABILITY** problem was given:

- The variables of φ and their negations form the vertices of $G(\varphi)$.
- There is an arc (α, β) iff there is a clause $\bar{\alpha} \vee \beta$ or $\beta \vee \bar{\alpha}$ in φ , where $\bar{\alpha}$ is the complement of α , i.e.: If α is true in some satisfying assignment \mathcal{I} of φ , then β must also be true in \mathcal{I} .
- It can be shown that φ is unsatisfiable iff there is a variable x such that there are paths from x to $\neg x$ and from $\neg x$ to x in $G(\varphi)$.

Notation. It is convenient to write $x \Rightarrow y$ if y is reachable from x in the graph $G(\varphi)$.

Exercise 1 (4 credits) Give a rigorous proof of the “if”-direction of the correctness of the above reduction, i.e.:

If there exists a variable x , s.t. $x \Rightarrow \neg x$ and $\neg x \Rightarrow x$ in $G(\varphi)$, then φ is unsatisfiable.

Hint. Carefully distinguish between what is assumed, what is defined and what has to be shown. The proof could thus start as follows:

Proof. (indirect) Suppose that there exists a variable x , s.t. $x \Rightarrow \neg x$ and $\neg x \Rightarrow x$ in $G(\varphi)$. Moreover, suppose that there exists a model \mathcal{I} of φ . We derive a contradiction by showing that then both x and $\neg x$ are true in \mathcal{I} .

For the “only if”-direction of the correctness proof of the problem reduction from **co-2-SAT** to **REACHABILITY**, we have to show the following implication: If there exists no variable x , s.t. $x \Rightarrow \neg x$ and $\neg x \Rightarrow x$ in $G(\varphi)$, then there exists a model \mathcal{I} of φ . To this end, consider the truth assignment \mathcal{I} constructed by the following algorithm:

```
/* Step 1 */  
for each literal  $x$ , s.t.  $\bar{x} \Rightarrow x$  do  
   $\mathcal{I}(x) := \mathbf{true}$ ; /* hence, implicitly,  $\mathcal{I}(\bar{x}) := \mathbf{false}$ ; */
```

```

    for each literal  $z$  with  $x \Rightarrow z$  do  $\mathcal{I}(z) := \mathbf{true}$  od;
od;
/* Step 2 */
while there exists  $x \in \text{Var}(\varphi)$ , s.t.  $\mathcal{I}(x)$  is undefined do
    choose an arbitrary variable  $x$ , s.t.  $\mathcal{I}(x)$  is undefined;
     $\mathcal{I}(x) := \mathbf{true}$ ;
    for each literal  $z$  with  $x \Rightarrow z$  do  $\mathcal{I}(z) := \mathbf{true}$  od;
od;

```

To show that an assignment \mathcal{I} thus constructed is indeed a model of φ , it is convenient to proof the following lemmas.

Lemma 1 *Suppose that there exists no variable x , s.t. both $x \Rightarrow \bar{x}$ and $\bar{x} \Rightarrow x$ hold. Then a truth assignment made by the above algorithm is never changed later, i.e.: it cannot happen, that at some stage, $\mathcal{I}(z) = \mathbf{true}$ for some variable z and later this value is changed to $\mathcal{I}(z) = \mathbf{false}$ or vice versa.*

Lemma 2 *Suppose that there exists no variable x , s.t. both $x \Rightarrow \bar{x}$ and $\bar{x} \Rightarrow x$ hold. Then the truth assignment \mathcal{I} constructed by the above algorithm is a model of φ .*

Exercise 2 (4 credits) Give a rigorous proof of Lemma 1.

Exercise 3 (2 credits) Give a rigorous proof of Lemma 2.