First Exercise Sheet for Compex Analysis

1. Calculate:
$$\sqrt[3]{1-i},\ \sqrt[n]{-1},\ \sqrt[n]{i} \quad \text{for } n = 1, 2, \ldots$$
(with sketch!)

2. Display graphically:
$\{z \in \mathbb{C} : |z - (1 - 3i)| \leq 2\}$; $\{z \in \mathbb{C} : \mathrm{Im}((z - i)(1 + i)) > 0\}$;
$\{z \in \mathbb{C} : |z - 1| < |z + 3|\}$; $\{z \in \mathbb{C} : |z| + |z + i| = 2\}$.

3. Let $|z - 1| = 1$. Show:
$$\arg(z - 1) = 2 \arg z = \frac{2}{3} \arg(z^2 - z)$$

4. Let $\mathbb{R}[x]$ be the ring of polynomials with real coefficients. Define an equivalence relation $\sim$ on $\mathbb{R}[x]$ via
$$P_1(x) \sim P_2(x) \iff P_1(x) - P_2(x) \text{ is divisible by } x^2 + 1 \text{ without remainder.}$$
The set of thus defined equivalence classes, together with the operations + und $\cdot$ defined in the usual sense, form the quotient ring $\mathbb{R}[x]/(x^2 + 1)$. Introduce an isomorphism between $\mathbb{R}[x]/(x^2 + 1)$ and $\mathbb{C}$. (This model of $\mathbb{C}$ was supplied by Cauchy.)

5. Where are the following functions differentiable?
$$\begin{aligned} f(z) &= (\bar{z})^2 \\ f(z) &= \log \sqrt{x^2 + y^2} + i \arctan \tfrac{y}{x} \\ f(z) &= \sin^2(x + y) + i \cos^2(x + y) \end{aligned}$$
Here, $z = x + i\,y$.

6. Show: A holomorphic function on a connected domain is, up to addition by a (real) constant, uniquely determined by its imaginary part.

7. Let $u : \mathbb{C} \to \mathbb{R}$ defined by
$$u(x, y) = x^2 - y^2 + e^{-y} \sin x - e^y \cos x.$$
Determine all functions $v : \mathbb{C} \to \mathbb{R}$, such that u and v satisfy the Cauchy-Riemann differential equations on all of $\mathbb{C}$.

8. Show: If f is holomorphic on a domain G and it holds that $|f|$ =const., then f is constant on all of G.