Examples for the test on Optimization, WS 2012

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Theoretical questions. One of the following theoretical tasks will be given.

1. (Part of the Ljusternik theorem.) Let

$$K := \{x \in \mathbf{R}^n : \; g(x) = 0, \; h(x) \le 0\},$$

where $g : \mathbf{R}^n \mapsto \mathbf{R}^m$, $h : \mathbf{R}^n \mapsto \mathbf{R}^r$ are differentiable at $x \in K$. Prove that

$$T_K(x) \subset \left\{ l \in \mathbf{R}^n : \; \partial g(x)\, l = 0, \; \partial h(x)_{|J(x)}\, l \le 0 \right\}.$$

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2. (Part of the Farkas lemma.) Let

$$P = \{x \in \mathbf{R}^n : \; Gx = 0, \; Hx \le 0\},$$

where G is an $(m \times n)$-matrix and H is an $(r \times n)$-matrix. Prove that the polar cone P° satisfies the inclusion

$$P^\circ \subset \{G'\lambda + H'\mu : \; \lambda \in \mathbf{R}^m, \; \mu \in \mathbf{R}^r, \; \mu \ge 0\}.$$

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3. Formulate the KKT theorem for general differentiable (not necessarily convex) functions. Is the normal form of the theorem (with $\lambda_0 \neq 0$) true for the example

$$\min\{x_1\}$$

subject to

$$-(x_1)^3 + x_2 \leq 0$$

$$-x_2 \leq 0.$$

Solution.

The problem is

$$\min_{x \in K} f(x), \tag{1}$$

where

$$K = \{x \in \mathbf{R}^n : \; g(x) = 0, \; h(x) \leq 0\}, \tag{2}$$

and $g : \mathbf{R}^n \mapsto \mathbf{R}^m$, $h : \mathbf{R}^n \mapsto \mathbf{R}^r$.

Define $J(x) = \{j : \; h_j(x) = 0\}$.

Theorem. *Let $x^* \in K$ be a local solution of problem (1), (2). Assume that the functions f, g and h are continuously differentiable around x^*.*

Then

(i) there exists a non-zero vector $(\lambda_0, \lambda, \mu) \in \mathbf{R} \times \mathbf{R}^m \times \mathbf{R}^r$ such that

$$\lambda_0 \partial f(x^*) + \lambda' \partial g(x^*) + \mu' \partial h(x^*) = 0, \tag{3}$$

$$\mu_j h_j(x^*) = 0, \quad j = 1, \ldots, r, \tag{4}$$

$$\mu \geq 0, \quad \lambda_0 \geq 0. \tag{5}$$

(ii) If, in addition, the matrix

$$\begin{pmatrix} \partial g(x^*) \\ \partial h(x^*)_{|J(x^*)} \end{pmatrix}$$

is surjective (that is, its rank is $m + |J(x^*)|$, where $|J(x^*)|$ is the number of the elements of $J(x^*)$), then claim (i) of the theorem is true with $\lambda_0 = 1$.

Answer to the additional question.

The point $(0,0)'$ satisfies the constraints. For any other admissible point $(x_1, x_2)'$ we have $0 \leq x_2 \leq (x_1)^3$, hence $x_1 \geq 0$. Then $(0,0)'$ is a solution of the problem. Equation (3) reads in this case as

$$\lambda_0 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \mu_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \mu_2 \begin{pmatrix} 0 \\ -1 \end{pmatrix} = 0,$$

which is possible only if $\lambda_0 = 0$. Thus the normal form of the KKT does not hold in this case.

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4. Formulate the KKT theorem for convex problems of the type

$$\min_{K \cap K_0} f(x) \tag{6}$$

where $K \subset \mathbf{R}^n$ is defined by the constraints

$$G_i x - b_i = 0, \ \ i = 1, \dots, m, \tag{7}$$

$$H_j x - c_j \leq 0, \ \ j = 1, \dots, r', \tag{8}$$

$$h_j(x) \leq 0, \ \ j = r'+1, \dots, r, \tag{9}$$

$0 \leq r' \leq r$, and K_0 is a polyhedral set.

Is the Slater condition fulfilled for the following sets (**the sets may be changed in the test!**)

$$K := \{x \in \mathbf{R}^3 : \ x_1 - x_3 = 0, \ x_2 - x_3 \leq -1, \ (x_1)^2 + (x_2)^2 \leq 1\},$$

$$K_0 := \{x \in \mathbf{R}^3 : \ x_2 \geq 0\}.$$

Solution.

Theorem. *Assume that the functions f and h_j are convex and that there exists $\tilde{x} \in K_0$ satisfying (7) and (8), and also satisfying (9) as strict inequalities (the Slater condition).*

Then x^* is an optimal solution if and only if $x^* \in K \cap K_0$ and there exist $\lambda \in \mathbf{R}^m$ and $\mu \in \mathbf{R}^r_+$ such that

$$L(x^*, \lambda, \mu) \leq L(x, \lambda, \mu) \quad \forall\, x \in K_0, \quad \textit{and} \ \ \mu' h(x^*) = 0, \tag{10}$$

where $L(x, \lambda, \mu) = f(x) + \lambda' g(x) + \mu' h(x)$.

The example: If $x \in K \cap K_0$, then $x_3 = x_1$, hence $x_2 - x_1 \leq -1$. Since $x_2 \geq 0$ we have $x_1 \geq 1 + x_2 \geq 1$ and from $(x_1)^2 + (x_2)^2 \leq 1$ we obtain that $x_1 = 1$ and $x_2 = 0$. This is the only point in $K \cap K_0$ and the non-linear constrain is satisfied as an equality. Thus the Slater condition is not fulfilled. (This conclusion is also evident from geometric representation on the (x_1, x_2)-plane.)

Remark. If the non-linear constraint were $(x_1)^2 + (x_2)^2 \leq 1.0001$, the Slater condition would be fulfilled with $\tilde{x} = (1, 0, 1)$, for example.

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5. Formulate the dual problem to the following linear optimization problem

$$\min \langle c, x \rangle$$

$$\begin{aligned} A_i x &= b_i, \quad i = 1, \ldots, m, \\ A_i x &\geq b_i, \quad i = m+1, \ldots, m+r, \end{aligned}$$

where $c \in \mathbf{R}^n$, A_i are n-dimensional row-vectors and b_i are real numbers.

One of the claims of the duality theorem 4.3 is:
if both the primal and the dual problem have feasible points, then they have optimal solutions. Prove this.

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6. Formulate the dual problem to the following linear optimization problem

$$\min \langle c, x \rangle$$

$$\begin{aligned} A_i x &\geq b_i, \quad i = 1, \ldots, r, \\ x_k &\geq 0, \quad k = 1, \ldots, l, \end{aligned}$$

where $c \in \mathbf{R}^n$, A_i are n-dimensional row-vectors and b_i are real numbers.

One of the claims of the duality theorem 4.3 applied to this problem is:
(B4) x^* and y^* are feasible points for GLP and GLD, respectively, and the complementary slackness conditions hold:

$$y_j^* \, (A_j x^* - b_j) = 0, \; j = 1, \ldots, r, \qquad x_k^* \, ([A']_k y^* - c_k) = 0, \; k = 1, \ldots, l.$$

(B3) (x^*, y^*) is a saddle point of the Lagrange function in $K_0 \times Y_0$;

Give a proof of this claim.

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7. Formulate and prove the general duality theorem (Theorem 3.18).

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8. Prove the following statement (a part of Theorem 4.10 in the script):

Theorem. *Let A be a non-zero matrix. If the linear problem*

$$\min_{x\in K}\{c'x\}, \qquad K=\{x\in \mathbf{R}^n:\ Ax=b,\ x\geq 0\},$$

has a feasible point (that is, $K\neq\emptyset$), then it has a basis point.

Samples of particular problems. Three problems similar to the following ones will be given in addition to the theoretical task. (The problems will be individual for every student!)

Problem 1. Solve the problem

$$\min\{2x_1+x_2+2x_3+x_4\}$$

subject to

$$\begin{array}{rcrcrcrcr} x_1 & + & x_2 & + & 5x_3 & + & x_4 & = & 7 \\ & & 2x_2 & + & 7x_3 & + & x_4 & = & 11, \end{array}$$

$$x_1,\, x_2,\, x_3,\, x_4 \geq 0.$$

by using the simplex method.

Hint: Use x_1 and one additional variable z_5 in the second equation as initial basis variables in the auxiliary problem for finding an initial basis for the given problem.

Solution. The auxiliary problem for finding an initial basis point is

$$\min\{z_5\}$$

subject to

$$\begin{array}{rrcrcrcrcr} x_1 & & + & x_2 & + & 5x_3 & + & x_4 & = & 7 \\ & z_5 & + & 2x_2 & + & 7x_3 & + & x_4 & = & 11, \end{array}$$

$$x_1,\ x_2,\ x_3,\ x_4,\ z_5 \geq 0.$$

We have (with $c = (0,0,0,0,1)'$)

$$\Delta_2 = c_2 - (c_1.1 + c_5.2) = -2,$$

$$\Delta_3 = c_3 - (c_1.5 + c_5.7) = -7,$$

$$\Delta_4 = c_4 - (c_1.1 + c_5.1) = -1.$$

We can chose any of the non-basis variables as a new basis variable. Let us take $\nu = 3$ (since Δ_2 is the minimal). Then according to the rule for choosing which basis variable to be removed we have to choose $\kappa = 1$, since

$$\frac{7}{5} < \frac{11}{7}.$$

Then we have

$$\begin{array}{ccccccccc} x_3 & & + & \frac{1}{5}x_1 & + & \frac{1}{5}x_2 & + & \frac{1}{5}x_4 & = & \frac{7}{5} \\ & z_5 & + & 2x_2 & + & 7\left(\frac{7}{5} - \frac{1}{5}x_1 - \frac{1}{5}x_2 - \frac{1}{5}x_4\right) & + & x_4 & = & 11, \end{array}$$

Hence, the adapted to the basis $(x_{,}\, z_5)$ representation of the equality constraints is

$$\begin{array}{cccccccccc} x_3 & & + & \frac{1}{5}x_1 & + & \frac{1}{5}x_2 & + & \frac{1}{5}x_4 & = & \frac{7}{5} \\ & z_5 & - & \frac{7}{5}x_1 & + & \frac{3}{5}x_2 & - & \frac{2}{5}x_4 & = & \frac{6}{5}, \end{array}$$

Then we calculate (still with $c = (0,0,0,0,1)'$)

$$\Delta_1 = c_1 - (c_3.\frac{1}{5} + c_5.\frac{-7}{5}) = \frac{7}{5},$$

$$\Delta_2 = c_2 - (c_3.\frac{1}{5} + c_5.\frac{3}{5}) = -\frac{3}{5},$$

$$\Delta_4 = c_4 - (c_3.\frac{1}{5} + c_5.\frac{-2}{5}) = \frac{2}{5}.$$

According to the rule of the simplex algorithm we have to choose $\nu = 2$, and since

$$\frac{7/5}{1/5} > \frac{6/5}{3/5}$$

we have to choose $\kappa = 5$. That is, z_5 leaves the basis and x_2 enters the basis.

We express

$$\begin{array}{ccccccccc} x_2 & & - & \frac{7}{3}x_1 & - & \frac{2}{3}x_4 & + & \frac{5}{3}z_5 & = & 2 \\ x_3 & + & \frac{1}{5}x_1 & + & \frac{1}{5}\left(2 - \frac{7}{3}x_1 - \frac{2}{3}x_4 - \frac{5}{3}z_5\right) & + & \frac{1}{5}x_4 & = & \frac{7}{5}, \end{array}$$

Since all the auxiliary variables have left the basis (this is only z_5 in our problem and we ignore it further) we have found an initial basis for the original problem, $((x_2,\, x_3))$, and the adapted representation, found from the above equations, is

$$\begin{array}{cccccccc} x_2 & & - & \frac{7}{3}x_1 & - & \frac{2}{3}x_4 & = & 2 \\ & x_3 & + & \frac{2}{3}x_1 & + & \frac{1}{3}x_4 & = & 1. \end{array}$$

Now we calculate (this time with $c = (2,1,2,1)'$)

$$\Delta_1 = c_1 - (c_2.\frac{-7}{3} + c_3.\frac{2}{3}) = 3,$$

$$\Delta_4 = c_4 - (c_2.\frac{-2}{3} + c_3.\frac{1}{3}) = 1.$$

Since Δ_1 and Δ_4 are both positive, we have reached an optimal solution, namely, $x^* = (0,2,1,0)'$.

Problem 2. Describe analytically the tangent and the normal cone to the set $K \subset \mathbf{R}^3$ defined by the constraints

$$(x_1)^2 + 2(x_3)^2 \leq 3,$$

$$(x_1)^3 + (x_2)^3 = 9$$

at the point $x^* = (1,2,1)' \in K$.

Hint: Check if the assumptions of the Ljusternik theorem are fulfilled and apply it to find $T_K(x^*)$. Then apply the Farkas lemma to find $N_K(x^*)$.

Solution. Here

$$g(x) = (x_1)^3 + (x_2)^3 - 9, \qquad h(x) = (x_1)^2 + 2(x_3)^2 - 3.$$

Then

$$\begin{aligned} \partial g(x^*) &= \left(3(x_1^*)^2,\, 3(x_2^*)^2,\, 0\right) = (3,12,0), \\ \partial h(x^*) &= (2x_1^*,\, 0,\, 4x_3^*) = (2,0,4). \end{aligned}$$

For $\bar{l} := (4, -1, -3)'$ (this is just one of many possible choices) we have

$$\partial g(x^*)\,\bar{l} = (3, 12, 0)\begin{pmatrix} 4 \\ -1 \\ -3 \end{pmatrix} = 0, \qquad \partial h(x^*)\,\bar{l} = (2, 0, 4)\begin{pmatrix} 4 \\ -1 \\ -3 \end{pmatrix} = -4 < 0,$$

thus the assumptions of the Ljusternik theorem are fulfilled.

Then

$$\begin{aligned} T_K(x^*) &= \left\{ l \in \mathbf{R}^3 : \ (3, 12, 0)\begin{pmatrix} l_1 \\ l_2 \\ l_3 \end{pmatrix} = 0, \quad (2, 0, 4)\begin{pmatrix} l_1 \\ l_2 \\ l_3 \end{pmatrix} \le 0 \right\} \\ &= \{ l \in \mathbf{R}^3 : \ 3l_1 + 12l_2 = 0, \ 2l_1 + 4l_3 \le 0 \}. \end{aligned}$$

Then we apply the Farkas lemma with $G = (3, 12, 0)$ and $H = (2, 0, 4)$ and obtain

$$\begin{aligned} N_K(x^*) &= \left\{ \begin{pmatrix} 3 \\ 12 \\ 0 \end{pmatrix} \lambda \ + \begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix} \mu : \ \lambda \in \mathbf{R}, \ \mu \ge 0 \right\} \\ &= \left\{ \begin{pmatrix} 3\lambda + 2\mu \\ 12\lambda \\ 4\mu \end{pmatrix} : \ \lambda \in \mathbf{R}, \ \mu \ge 0 \right\}. \end{aligned}$$

Problem 3. Consider the problem

$$\min_{x \in K} \{ f(x) := x_1 - 2x_2 + 3x_3 - 4x_4 \}$$

with

$$K := \left\{ x \in \mathbf{R}^4 : \ (x_1)^2 + (x_2)^2 + (x_3)^2 + (x_4)^2 \le 2 \right\}$$

as a primal problem and find the function $D(\mu)$ of the corresponding dual problem

$$\max_{\mu \ge 0} D(\mu).$$

Then find the solution μ^* of the dual problem and evaluate the corresponding x^* (resulting from the definition of $D(\mu^*)$). Is x^* a solution of the primal problem?

Solution. Here

$$L(x, \mu) = x_1 - 2x_2 + 3x_3 - 4x_4 + \mu\left((x_1)^2 + (x_2)^2 + (x_3)^2 + (x_4)^2 - 2\right). \qquad (11)$$

Since by definition $D(\mu) = \min_{x \in \mathbf{R}^4} L(x, \mu)$ and L is convex with respect to x the minimizing x is determined by the condition $\partial_x L(x, \mu) = 0$:

$$x_1 = -\frac{1}{2\mu}, \quad x_2 = \frac{1}{\mu}, \quad x_3 = -\frac{3}{2\mu}, \quad x_4 = \frac{2}{\mu}. \tag{12}$$

Then substituting in (11) we calculate

$$D(\mu) = -\frac{15}{2\mu} - 2\mu.$$

Since $D(\mu)$ is concave, its maximum on the set $\mu \geq 0$ is attained either for $\mu = 0$ (which gives the "bad" value $D = -\infty$, which cannot be maximal), or at a point where $\partial D(\mu) = 0$:

$$\frac{15}{2\mu^2} - 2 = 0$$

hence (taking into account that $\mu \geq 0$)

$$\mu^* = \frac{\sqrt{15}}{2}.$$

We evaluate $D(\mu^*) = -2\sqrt{15}$.

From (12) we calculate a "candidate" for a solution of the primal problem:

$$x_1^* = -\frac{1}{\sqrt{15}}, \quad x_2^* = \frac{2}{\sqrt{15}}, \quad x_3^* = -\frac{3}{\sqrt{15}}, \quad x_4^* = \frac{4}{\sqrt{15}},$$

Then we evaluate

$$f(x^*) = \frac{1}{\sqrt{15}}(-1 - 4 - 9 - 16) = -2\sqrt{15} = D(\mu^*),$$

which means that x^* is an optimal solution due to the general duality theorem (Theorem 3.12 in the script).

Problem 4. Write down the dual problem to the following one:

$$\min\{2x_1 + x_2 + 4x_3\}$$

$$x_1 - x_2 + x_3 \geq 2$$

$$-2x_1 + x_2 + 2x_3 \geq 1$$

$$x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0.$$

Solve the dual problem geometrically and use the duality theorem to determine an optimal solution of the primal problem.

Solution. The dual problem reads as

$$\max\{2y_1 + y_2\}$$

$$y_1 - 2y_2 \leq 2$$

$$-y + y_2 \leq 1$$

$$y_1 + 2y_2 \leq 4$$

$$y_1 \geq 0,\ y_2 \geq 0.$$

The graphical solution (I skip it here) gives $y_1^* = 3$, $y_2^* = \frac{1}{2}$ and the second inequality is non-active. The complementary slackness condition in the duality theorem implies that $x_2^* = 0$. Since y_1^* and y_2^* are both positive, the complementary slackness condition implies also that the two inequality constraints in the primal problem are active. Taking into account that $x_2^* = 0$ we have the equations

$$\begin{aligned} x_1 + x_3 &= 2 \\ -2x_1 + 2x_3 &= 1. \end{aligned}$$

Solving them we obtain the following solution of the primal problem: $x^* = \left(\frac{3}{4}, 0, \frac{5}{4}\right)'$.

Problem 5. Solve the problem

$$\min\{(x_1)^2 + 2(x_2)^2 + (x_3)^2\}$$

$$\begin{aligned} x_1 + x_2 - \ln x_3 &\geq 1 \\ x_3 &\geq 1. \end{aligned}$$

by using the KKT theorem.

Solution. First we reformulate the problem in the form as in the KKT theorem:

$$\min\{(x_1)^2 + 2(x_2)^2 + (x_3)^2\}$$

$$-x_1 - x_2 + \ln x_3 + 1 \leq 0 \quad (13)$$

$$-x_3 + 1 \leq 0. \quad (14)$$

This problem has a solution since the level sets $\{x \in \mathbf{R}^3 : f(x) \leq c\}$ of the objective function are compact and the existence theorem from Chapter 1 is applicable. Every optimal solution is a part of a KKT point.

First we search for KKT points with $\lambda_0 = 1$. The Lagrange function is

$$L(x, \mu) = (x_1)^2 + 2(x_2)^2 + (x_3)^2 + \mu_1(-x_1 - x_2 + \ln x_3 + 1) + \mu_2(-x_3 + 1)$$

and the KKT conditions consist of the equations

$$\partial_{x_1} L = 2x_1 - \mu_1 = 0 \quad (15)$$

$$\partial_{x_2} L = 4x_2 - \mu_1 = 0 \quad (16)$$

$$\partial_{x_3} L = 2x_3 + \frac{\mu_1}{x_3} - \mu_2 = 0 \quad (17)$$

$$\mu_1(-x_1 - x_2 + \ln x_3 + 1) = 0 \quad (18)$$

$$\mu_2(-x_3 + 1) = 0 \quad (19)$$

and the inequalities (13), (14) and $\mu_1 \geq 0$, $\mu_2 \geq 0$.

We consider the following four cases.

(i) $\mu_1 = \mu_2 = 0$. Then from (15)–(17) $x_1 = x_2 = x_3 = 0$, which is not a feasible point due to (14).

(ii) $\mu_1 = 0$, $\mu_2 > 0$. Then from (19) $x_3 = 1$ and from (15), (16) $x_1 = x_2 = 0$. This is not a feasible point due to (13).

(iii) $\mu_1 > 0$, $\mu_2 = 0$. Then (15)–(18) give

$$2x_1 - \mu_1 = 0$$

$$4x_2 - \mu_1 = 0$$

$$2x_3 + \frac{\mu_1}{x_3} = 0$$

$$x_1 + x_2 - \ln x_3 = 1.$$

Then $x_1 = 2x_2$ from the first two equations, $4x_2 = \mu_1$ from the second equation, and $\mu_1 = -2(x_3)^2$ from the third equation. Then the fourth equation takes the form

$$-\frac{3}{2}(x_3)^2 - \ln x_3 = 1.$$

Then $2 - \ln x_3$ must be negative, which is not the case since $x_3 \geq 1$. Thus we do not obtain a KKT point also in this case.

(iv) $\mu_1 > 0$, $\mu_2 > 0$. From (19) we get $x_3 = 1$. Equations (15), (16) implay $x_1 = 2x_2$ and (18) implies $x_1 + x_2 = 1$. Then $x_1 = \frac{2}{3}$, $x_2 = \frac{1}{3}$. So we obtain a KKT point with $x = (\frac{2}{3}, \frac{1}{3}, 1)'$.

Now we try to find abnormal KKT points (with $\lambda_0 = 0$). Such may arise only if the surjectivity condition in the KKT theorem is not fulfilled. We have

$$\partial h(x) = \begin{pmatrix} -1 & -1 & \frac{1}{x_3} \\ 0 & 0 & -1 \end{pmatrix},$$

which has rank $= 2$ for every $x_3 \geq 1$. Thus the surjectivity condition is fulfilled and there are no abnormal KKT points.

Since the only KKT point is $x = (\frac{2}{3}, \frac{1}{3}, 1)'$ and since the problem has a solution, this point is the unique optimal solution.