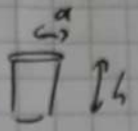
Tut EPhy 1 UE1 SS24

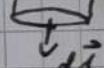
1, $\vec{F}(\vec{r}) = \begin{pmatrix} 4x \\ -2y^2 \\ z^2 \end{pmatrix}$  Zyl. Koord: $x = r\cos\varphi$
$y = r\sin\varphi$
$z = z$

a, $\vec{\nabla}\vec{F} = \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix} \cdot \begin{pmatrix} 4x \\ -2y^2 \\ z^2 \end{pmatrix} = 4 - 4y + 2z$

$= 4 - 4r\sin\varphi + 2z$

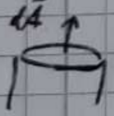
$$\int_V \vec{\nabla}\vec{F}\, d^3x = \int_0^h dz \int_0^{2\pi} d\varphi \int_0^a r\,dr \,(4 - \underbrace{4r\sin\varphi}_{\to\emptyset} + 2z) =$$

$$= 4\cdot\pi a^2 h + 2\cdot 2\pi\cdot\frac{a^2}{2}\frac{h^2}{2} = 4\pi a^2 h + \pi a^2 h^2$$

b, Boden: $\vec{F}(z=0) = \begin{pmatrix} 4x \\ -2y^2 \\ 0 \end{pmatrix}$  $d\vec{A} \parallel \hat{e}_z \Rightarrow \vec{F}(z=0) \perp d\vec{A}$

$\Rightarrow \int_{Boden} \vec{F} d\vec{A} = \emptyset$

Deckel: $\vec{F}(z=h) = \begin{pmatrix} 4x \\ -2y^2 \\ h^2 \end{pmatrix}$



$d\vec{A} = r\,dr\,d\varphi\, \hat{e}_z$

$$\Rightarrow \int_{Deckel} \vec{F} d\vec{A} = \int_0^{2\pi} d\varphi \int_0^a r\,dr \begin{pmatrix} 4x \\ -2y^2 \\ h^2 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} =$$

$$= h^2 \cdot 2\pi \cdot \frac{a^2}{2} = \pi a^2 h^2$$

Mantel: $\vec{F}(r=a) = \begin{pmatrix} 4a\cos\varphi \\ -2a^2\sin^2\varphi \\ z^2 \end{pmatrix}$

$$d\vec{A} = \frac{\partial\vec{r}}{\partial\varphi} \times \frac{\partial\vec{r}}{\partial z}\, d\varphi\, dz = \partial_\varphi \begin{pmatrix} r\cos\varphi \\ r\sin\varphi \\ z \end{pmatrix} \times \partial_z \begin{pmatrix} r\cos\varphi \\ r\sin\varphi \\ z \end{pmatrix} d\varphi\, dz =$$

$$= \begin{pmatrix} -r\sin\varphi \\ r\cos\varphi \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} d\varphi\, dz = \begin{pmatrix} r\cos\varphi \\ r\sin\varphi \\ 0 \end{pmatrix} d\varphi\, dz \underset{r=a}{=} \begin{pmatrix} a\cos\varphi \\ a\sin\varphi \\ 0 \end{pmatrix} d\varphi\, dz$$

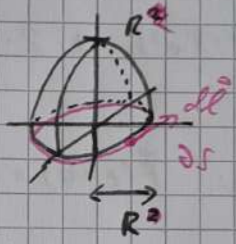
$$\int_{\text{Mantel}} \vec{F}\, d\vec{A} = \int_0^{2\pi} d\varphi \int_0^h dz\, a \begin{pmatrix} 4a\cos\varphi \\ -2a^2\sin^2\varphi \\ z^2 \end{pmatrix} \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ 0 \end{pmatrix} =$$

$$= a \int_0^{2\pi} d\varphi \int_0^h dz \left(4a \underbrace{\cos^2\varphi}_{\pi} - 2a^2 \underbrace{\sin^3\varphi}_{\emptyset} \right) = 4\pi a^2 h$$

$$\Rightarrow \oint_{\partial V} \vec{F}\, d\vec{A} = 4\pi a^2 h + \pi a^2 h^2 \quad ✓$$

2) $\vec{F} = \begin{pmatrix} \sqrt{x^2+y^2} \\ y^2 \\ z^2 \end{pmatrix}$ $\quad S: \; z = R^2 - x^2 - y^2, \; z \geq 0, \; R \geq 0$

Zylkoord: $\vec{F} = \begin{pmatrix} r \\ r^2\sin^2\varphi \\ z^2 \end{pmatrix}$



$z=0:$

$0 = R^2 - r^2$

$\Rightarrow r = R$

a) $\oint_{\partial S} \vec{F}\, d\vec{l} \qquad d\vec{l} = \frac{\partial \vec{r}}{\partial \varphi} d\varphi = \partial_\varphi \begin{pmatrix} R\cos\varphi \\ R\sin\varphi \\ 0 \end{pmatrix} = R \begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix}$

$$\Rightarrow \oint_{\partial S} \vec{F}\, d\vec{l} = \int_0^{2\pi} d\varphi\, R \begin{pmatrix} R \\ R^2\sin^2\varphi \\ 0 \end{pmatrix} \begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix} =$$

$$= R \int_0^{2\pi} d\varphi \left(-R \underbrace{\sin\varphi}_{=0} + R^2 \underbrace{\sin^2\cos\varphi}_{=0} \right) = \emptyset$$

b) $\int_S \vec{\nabla} \times \vec{F}\, d\vec{A} \qquad \vec{\nabla} \times \vec{F} = \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix} \times \begin{pmatrix} \sqrt{x^2+y^2} \\ y^2 \\ z^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -\frac{y}{\sqrt{x^2+y^2}} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -\sin\varphi \end{pmatrix}$

$$d\vec{A} = \frac{\partial \vec{r}}{\partial r} \times \frac{\partial \vec{r}}{\partial \varphi}\, dr\, d\varphi = \partial_r \begin{pmatrix} r\cos\varphi \\ r\sin\varphi \\ z \end{pmatrix} \times \partial_\varphi \begin{pmatrix} r\cos\varphi \\ r\sin\varphi \\ z \end{pmatrix} dr\, d\varphi \underset{z = R^2 - r^2}{=} \partial_r \begin{pmatrix} r\cos\varphi \\ r\sin\varphi \\ R^2 - r^2 \end{pmatrix} \times \partial_\varphi \begin{pmatrix} r\cos\varphi \\ r\sin\varphi \\ R^2 - r^2 \end{pmatrix} dr\, d\varphi$$

$$= \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ -2r \end{pmatrix} \times \begin{pmatrix} -r\sin\varphi \\ r\cos\varphi \\ 0 \end{pmatrix} dr\, d\varphi = \begin{pmatrix} 2r^2\cos\varphi \\ 2r^2\sin\varphi \\ r \end{pmatrix} dr\, d\varphi$$

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2b) $\int_S \vec{\nabla} \times \vec{F} \, d\vec{f} = \int_0^{2\pi} d\varphi \int_0^{R} dr \begin{pmatrix} 0 \\ 0 \\ -\sin\varphi \end{pmatrix} \begin{pmatrix} 2r^2\cos\varphi \\ 2r^2\sin\varphi \\ r \end{pmatrix} =$

$= \int_0^{2\pi} d\varphi \int_0^{R} dr \, (-r\underbrace{\sin\varphi}_{=0}) = \emptyset$ ✓

3. Indexgymnastik

a) $$\vec{X} \equiv \vec{a} \times (\vec{b} \times \vec{c})$$

$$\varepsilon_{ijk}\, a_j\, \varepsilon_{klm}\, b_l\, c_m = \underbrace{\varepsilon_{ijk}\varepsilon_{klm}}\, a_j b_l c_m$$

$\det(A)\cdot\det(B) = \det(AB)$ $$\det\begin{pmatrix} | & | & | \\ e_i & e_j & e_k \\ | & | & | \end{pmatrix} \det\begin{pmatrix} | & | & | \\ e_k & e_l & e_m \\ | & | & | \end{pmatrix}$$

$\det(A) = \det(A^T)$ $$= \det\left(\begin{pmatrix} - & e_i & - \\ - & e_j & - \\ - & e_k & - \end{pmatrix} \begin{pmatrix} | & | & | \\ e_k & e_l & e_m \\ | & | & | \end{pmatrix} \right) =$$

$$\det\begin{pmatrix} \delta_{ik} & \delta_{il} & \delta_{im} \\ \delta_{jk} & \delta_{jl} & \delta_{jm} \\ \underbrace{\delta_{kk}}_{3} & \delta_{kl} & \delta_{km} \end{pmatrix} = 3 \cdot \det\begin{pmatrix} \delta_{il} & \delta_{im} \\ \delta_{jl} & \delta_{jm} \end{pmatrix} - \delta_{kl} \det\begin{pmatrix} \delta_{ik} & \delta_{im} \\ \delta_{jk} & \delta_{jm} \end{pmatrix} + \delta_{km} \det\begin{pmatrix} \delta_{ik} & \delta_{il} \\ \delta_{jk} & \delta_{jl} \end{pmatrix}$$

$$= 3(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) - \delta_{kl}\cdot(\delta_{ik}\delta_{jm} - \delta_{im}\delta_{jk}) + \delta_{km}(\delta_{ik}\delta_{jl} - \delta_{jk}\delta_{il})$$

$$= 3\delta_{il}\delta_{jm} - 3\delta_{im}\delta_{jl} - \delta_{il}\delta_{jm} + \delta_{im}\delta_{jl} + \delta_{im}\delta_{jl} - \delta_{il}\delta_{jm}$$

$$= \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$$

$$\rightarrow (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\, a_j b_l c_m = a_m b_i c_m - a_l b_l c_i$$

$$= b_i (a_m c_m) - (a_l b_l) c_i \rightarrow \vec{b}\cdot(\vec{a}\cdot\vec{c}) - (\vec{a}\cdot\vec{b})\cdot\vec{c}$$

$$\vec{X} \equiv \vec{a} \times (\vec{b} \times \vec{c}) = \underline{\vec{b}\,(\vec{a}\cdot\vec{c}) - \vec{c}\,(\vec{a}\cdot\vec{b})}$$

$\vec{a}, \vec{b}$ Vektorfelder abh. von $\vec{r} = x_i$

b)

Divergenz: $\vec{\nabla} \cdot (\vec{a} \times \vec{b}) = \partial_i \, \varepsilon_{ijk} \, a_j b_k =$

$$\varepsilon_{ijk} \, \partial_i (a_j b_k) = \varepsilon_{ijk} \left(b_k \partial_i a_j + a_j \partial_i b_k \right)$$

Produktregel

$$= b_k \, \varepsilon_{ijk} \, \partial_i a_j + a_j \, \varepsilon_{ijk} \, \partial_i b_k$$

$$= b_k \, \varepsilon_{kij} \, \partial_i a_j - a_j \, \varepsilon_{jik} \, \partial_i b_k$$

$$\rightarrow \underline{\vec{b} \cdot (\vec{\nabla} \times \vec{a}) - \vec{a} \cdot (\vec{\nabla} \times \vec{b})} = \underline{\vec{b} \cdot \mathrm{rot}(\vec{a}) - \vec{a} \, \mathrm{rot}(\vec{b})}$$

Rotation: $\vec{\nabla} \times (\vec{a} \times \vec{b}) = \varepsilon_{ijk} \, \partial_j \, \varepsilon_{klm} \, a_l b_m$

$$= \underbrace{\varepsilon_{ijk} \varepsilon_{klm}}_{\text{Grassmann-Identität}} \, \partial_j a_l b_m = (\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}) \, \partial_j a_l b_m$$

$$\varepsilon_{ijk} \varepsilon_{klm} = \delta_{il} \delta_{jm} - \delta_{im} \delta_{j}$$

$$= \partial_m a_i b_m - \partial_l a_l b_i$$

$$= \partial_m b_m a_i - \partial_l a_l b_i$$

$$= a_i \partial_m b_m + b_m \partial_m a_i - b_i \partial_l a_l - a_l \partial_l b_i$$

$$\rightarrow \vec{a} \, (\vec{\nabla} \cdot \vec{b}) + (\vec{b} \cdot \vec{\nabla}) \, \vec{a} - \vec{b} \, (\vec{\nabla} \cdot \vec{a}) - (\vec{a} \cdot \vec{\nabla}) \, \vec{b}$$

$$= \underline{\vec{a} \cdot \mathrm{div}(\vec{b}) + (\vec{b} \cdot \vec{\nabla}) \, \vec{a} - \vec{b} \, \mathrm{div}(\vec{a}) - (\vec{a} \cdot \vec{\nabla}) \, \vec{b}}$$

c) $\vec{r} = x_i \quad , \quad r = (x_i x_i)^{1/2} \qquad \vec{r}\,' \neq \vec{r}$

$$\vec{\nabla} \frac{1}{|\vec{r}-\vec{r}\,'|} = \partial_j \frac{1}{\left((x_i-x_i')(x_i-x_i')\right)^{1/2}} = -\frac{1}{2} \cdot \frac{1}{\left((x_l-x_l')(x_l-x_l')\right)^{3/2}} \cdot \partial_j \left((x_i-x_i')^2\right)$$

$$= \frac{(-1)}{\left((x_l-x_l')^2\right)^{3/2}} \cdot \left(\delta_{ij}\right) \cdot (x_i-x_i')$$

$$= \frac{(-1)(x_j-x_j')}{\left((x_l-x_l')^2\right)^{3/2}} = \frac{(-1)(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3} = \underline{\frac{-\vec{r}+\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3}}$$

$$\vec{\nabla} \frac{1}{|\vec{r}-\vec{r}\,'|} = -\frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3}$$ → wichtige Identität in der Elektrodynamik

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d) Umwandlung in Vektorschreibweise

$\varepsilon_{ijk}\, \partial_i B_k = -\varepsilon_{jik}\, \partial_i B_k = -\vec{\nabla} \times \vec{B} = \underline{-\operatorname{rot} \vec{B}}$

$e^2 = a_i a_j \delta_{ij} = a_i a_i = \underline{\vec{a} \cdot \vec{a}}$ (Skalarprodukt)

$f = \varepsilon_{abc}\, \partial_b \partial_c \phi = \vec{\nabla} \times (\vec{\nabla}\phi) = \underline{\operatorname{rot}(\operatorname{grad}(\phi))} = \underline{\emptyset}$

$h = \partial_m \delta_{km} E_i \delta_{ik} = \partial_m E_m = \vec{\nabla} \cdot \vec{E} = \underline{\operatorname{div} \vec{E}}$

e)

$$(\vec{a}\cdot\vec{\nabla})\left(\frac{\vec{r}}{r}f(r)\right) = a_i\,\partial_i\,\frac{x_j}{(x_k x_k)^{1/2}}\,f((x_l x_l)^{1/2})$$

$$a_i\left(\frac{f(\ldots)}{(x_k x_k)^{1/2}}\,\delta_{ij} + x_j f(\ldots)\,\partial_i\,\frac{1}{(x_k x_k)^{1/2}} + \frac{x_j}{(x_k x_k)^{1/2}}\,\partial_i f((x_l x_l)^{1/2})\right)$$

$$= \frac{a_j f((x_l x_l)^{1/2})}{(x_k x_k)^{1/2}} + x_j f((x_l x_l)^{1/2})\left(-\frac{1}{2}\right)\frac{1}{(x_k x_k)^{3/2}}\cdot 2\underbrace{\delta_{ik}x_k\cdot a_i}_{a_i x_i}$$

$$+ \frac{x_j}{(x_k x_k)^{1/2}}\,f'((x_k x_k)^{1/2})\,\frac{1}{2}\,\frac{1}{(x_l x_l)^{1/2}}\,2\cdot\overbrace{\delta_{li}\,x_l\cdot a_i}$$

$$= \vec{a}\cdot\frac{f(r)}{r} - \frac{f(r)}{r^3}\,\vec{r}\cdot(\vec{r}\cdot\vec{a}) + \frac{\vec{r}\,f'(r)}{r^2}\,(\vec{r}\cdot\vec{a})$$