5.1 Nie zu spät für Dualität

a)

Maxwellgleichungen im Vakuum ohne Quellen (in Gauß Einheiten)

$\mathrm{div}\,\vec{E} = 0$

$\mathrm{div}\,\vec{B} = 0$

$\mathrm{rot}\,\vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}$

$\mathrm{rot}\,\vec{B} = \frac{1}{c}\frac{\partial \vec{E}}{\partial t}$

$\vec{E} \to \vec{B}$

$\vec{B} \to -\vec{E}$

$\hookrightarrow \mathrm{div}\,\vec{B} = 0$

$\cdot\ \mathrm{div}\,(-\vec{E}) = 0$

$\hookrightarrow \mathrm{div}\,\vec{E} = 0$

$\mathrm{rot}\,\vec{B} = -\frac{1}{c}\frac{\partial(-\vec{E})}{\partial t} = \frac{1}{c}\frac{\partial \vec{E}}{\partial t}$

$-\mathrm{rot}\,\vec{E} = \frac{1}{c}\frac{\partial \vec{B}}{\partial t}$

$\to \mathrm{rot}\,\vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}$

ohne Quellen

$\Rightarrow$ Maxwellgl. im Vakuum sind invariant unter elektr. magn. Dualität

$\to$ gilt nicht bei Anwesenheit von Quellen

$\mathrm{div}\,\vec{B} = 0$ $\qquad\to\qquad$ $\mathrm{div}\,\vec{E} = 0$

$\mathrm{div}\,\vec{E} = 4\pi\rho$ $\qquad\qquad$ $\mathrm{div}\,\vec{B} = 4\pi\rho$

$\Rightarrow$ Problem! B-Feld immer Quellenfrei

b)

Maxwell Gleichungen mit magn. Ladungen / Strömen

$\mathrm{div}\,\vec{E} = 4\pi\rho_e$

$\mathrm{div}\,\vec{B} = 4\pi\rho_m$

$\mathrm{rot}\,\vec{E} = -\vec{j}_m\frac{4\pi}{c} - \frac{1}{c}\frac{\partial \vec{B}}{\partial t}$

$\mathrm{rot}\,\vec{B} = \frac{4\pi}{c}\vec{j}_e + \frac{1}{c}\frac{\partial \vec{E}}{\partial t}$

mit magn. Strömen/Quellen

Maxwell gl. inv. unter allgemeiner Dualität

c)

Transformationen

$\to$ Quelle u. Strom transformieren gleich mit

$\vec{E}' = \vec{E}\cos(\xi) - \vec{B}\sin(\xi)$

$\rho_e' = \rho_e\cos(\xi) - \rho_m\sin(\xi)$

$\vec{j}_e' = \vec{j}_e\cos(\xi) - \vec{j}_m\sin(\xi)$

$\vec{B}' = \vec{B}\cos(\xi) + \vec{E}\sin(\xi)$

$\rho_m' = \rho_m\cos(\xi) + \rho_e\sin(\xi)$

$\vec{j}_m' = \vec{j}_m\cos(\xi) + \vec{j}_e\sin(\xi)$

$\to$ Herleitung siehe nächste Seite

$\mathrm{div}\,\vec{E}' = 4\pi\rho_e'$

$\mathrm{div}\,\vec{B}' = 4\pi\rho_m'$

$\mathrm{rot}\,\vec{E}' = -\vec{j}_m'\frac{4\pi}{c} - \frac{1}{c}\frac{\partial \vec{B}'}{\partial t}$

$\mathrm{rot}\,\vec{B}' = \vec{j}_e'\frac{4\pi}{c} + \frac{1}{c}\frac{\partial \vec{E}'}{\partial t}$

$$\operatorname{div}(\vec{E}') = \cos(\xi)\,\underbrace{\operatorname{div}(\vec{E})}_{4\pi\rho_e} - \sin(\xi)\,\underbrace{\operatorname{div}(\vec{B})}_{4\pi\rho_m} = 4\pi\rho_e'$$

$$\underline{\rho_e' = \cos(\xi)\,\rho_e - \rho_m \sin(\xi)}$$

aus $\vec{E} \to \vec{B}$, $\vec{B} \to -\vec{E}$ folgt $\xi = \frac{3\pi}{2}$

bzw. $\xi = 0$ keine Veränderung

$$\underline{\vec{B}' = \vec{B}\cdot\cos(\xi) + \vec{E}\sin(\xi)}$$

$$\operatorname{div}(\vec{B}') = \cos(\xi)\,\underbrace{\operatorname{div}\vec{B}}_{4\pi\rho_m} + \sin(\xi)\,\underbrace{\operatorname{div}\vec{E}}_{4\pi\rho_e} = 4\pi\rho_m'$$

$$\underline{\rho_m' = \rho_m\cos(\xi) + \rho_e\sin(\xi)}$$

Transformationsmatrix

$$\begin{pmatrix}\vec{E}'\\ \vec{B}'\end{pmatrix} = \begin{pmatrix}\cos\xi & -\sin\xi\\ \sin\xi & \cos\xi\end{pmatrix}\begin{pmatrix}\vec{E}\\ \vec{B}\end{pmatrix}$$

↳ Rotationsmatrix

$$\operatorname{rot}\vec{E}' = -\vec{j}_m'\,\frac{4\pi}{c} - \frac{1}{c}\frac{\partial\vec{B}'}{\partial t}$$

$$\cos\xi\,\underbrace{\operatorname{rot}\vec{E}}_{-\vec{j}_m\frac{4\pi}{c} - \frac{1}{c}\frac{\partial\vec{B}}{\partial t}} - \sin\xi\,\underbrace{\operatorname{rot}\vec{B}}_{\vec{j}_e\frac{4\pi}{c} + \frac{1}{c}\frac{\partial\vec{E}}{\partial t}} = -\vec{j}_m'\,\frac{4\pi}{c} - \frac{1}{c}\frac{\partial\vec{B}'}{\partial t}$$

$$\hookrightarrow -\left[\cos\xi\,\vec{j}_m\frac{4\pi}{c} + \sin\xi\,\vec{j}_e\frac{4\pi}{c}\right] - \frac{1}{c}\left[\cos(\xi)\frac{\partial\vec{B}}{\partial t} + \sin(\xi)\frac{\partial\vec{E}}{\partial t}\right]$$

$$\underline{\vec{j}_m' = \cos(\xi)\,\vec{j}_m + \sin(\xi)\,\vec{j}_e}$$

analoge Berechnung für

$$\underline{\vec{j}_e' = \cos(\xi)\,\vec{j}_e - \sin(\xi)\,\vec{j}_m}$$

im SI System

Umrechnung
$$E = \sqrt{4\pi\varepsilon_0}\, E^{SI}$$
$$\rho_e = \frac{1}{\sqrt{4\pi\varepsilon_0}}\, \rho^{SI}$$
$$j_e = \frac{1}{\sqrt{4\pi\varepsilon_0}}\, j_e^{SI}$$
$$B = \sqrt{\frac{4\pi}{\mu_0}}\, B^{SI}$$
$$\rho_m = \sqrt{\frac{\mu_0}{4\pi}}\, \rho_m^{SI}, \quad j_m = \sqrt{\frac{\mu_0}{4\pi}}\, j_m^{SI}$$
$$\frac{1}{c} = \sqrt{\mu_0\varepsilon_0}$$

$$\operatorname{div}\vec{E}' = 4\pi \rho_e'$$

$$\sqrt{4\pi\varepsilon_0}\, \operatorname{div}\vec{E}'^{SI} = 4\pi \frac{\rho^{SI}}{\sqrt{4\pi\varepsilon_0}}$$

$$\operatorname{div}\vec{E}'^{SI} = \frac{\rho^{SI}}{\varepsilon_0}$$

$$\operatorname{rot}\vec{B}' = \frac{4\pi}{c}\vec{j_e}' + \frac{1}{c}\frac{\partial \vec{E}'}{\partial t}$$

$$\sqrt{\frac{4\pi}{\mu_0}}\, \operatorname{rot}\vec{B}^{SI\,\prime} = \frac{4\pi}{c}\frac{\vec{j_e}^{SI}}{\sqrt{4\pi\varepsilon_0}} + \frac{1}{c}\sqrt{4\pi\varepsilon_0}\,\frac{\partial\vec{E}^{SI\,\prime}}{\partial t}$$

$$\operatorname{rot}\vec{B}'^{SI} = \sqrt{\frac{\mu_0}{\varepsilon_0}}\sqrt{\mu_0\varepsilon_0}\,\vec{j_e}'^{SI} + \sqrt{\mu_0\varepsilon_0}\sqrt{\mu_0\varepsilon_0}\,\frac{\partial\vec{E}'^{SI}}{\partial t}$$

$$\operatorname{rot}\vec{B}'^{SI} = \mu_0\,\vec{j_e}'^{SI} + \mu_0\varepsilon_0\,\frac{\partial\vec{E}'^{SI}}{\partial t}$$

$$\operatorname{div}\vec{B}'^{SI} \cdot \sqrt{\frac{4\pi}{\mu_0}} = 4\pi\,\rho_m^{SI} \cdot \sqrt{\frac{\mu_0}{4\pi}}$$

$$\operatorname{div}\vec{B}'^{SI} = \rho_m^{SI} \cdot \mu_0$$

$$\operatorname{rot}\vec{E}'^{SI}\sqrt{4\pi\varepsilon_0} = -4\pi\sqrt{\mu_0\varepsilon_0}\cdot\sqrt{\frac{\mu_0}{4\pi}}\,\vec{j_m}'^{SI} - \sqrt{\mu_0\varepsilon_0}\sqrt{\frac{4\pi}{\mu_0}}\,\frac{\partial\vec{B}^{SI}}{\partial t}$$

$$\operatorname{rot}\vec{E}'^{SI} = -\mu_0\,\vec{j_m}'^{SI} - \frac{\partial\vec{B}'^{SI}}{\partial t}$$

Spezialfall $\xi = 0$:

$$R = \begin{pmatrix} \cos\xi & -\sin\xi \\ \sin\xi & \cos\xi \end{pmatrix}$$

$\vec{E} \to \vec{E}' = \vec{E}$
$\vec{B} \to \vec{B}' = \vec{B}$

$R(0) = \mathbb{1}$ → keine Veränderung

Spezialfall $\xi = \frac{\pi}{2}$: $R(\pi/2) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$\vec{E} \to \vec{E}' = -\vec{B}$
$\vec{B} \to \vec{B}' \to \vec{E}$

Transformation im SI-System

$$\vec{E}' = \vec{E}\cos(\xi) - \vec{B}\sin(\xi)\frac{1}{\sqrt{\varepsilon_0\mu_0}}$$

$$\vec{B}' = \vec{B}\cdot\cos(\xi) + \vec{E}\sin(\xi)\sqrt{\varepsilon_0\mu_0}$$

$$\mathrm{div}(\vec{E}') = \frac{\rho_e'}{\varepsilon_0}$$

$$\hookrightarrow \cos\underbrace{\mathrm{div}(\vec{E})}_{\frac{\rho_e}{\varepsilon_0}} - \sin\frac{1}{\sqrt{\varepsilon_0\mu_0}}\underbrace{\mathrm{div}(\vec{B})}_{\rho_m\cdot\mu_0}$$

$$= \frac{1}{\varepsilon_0}\left(\cos\rho_e - \sin\sqrt{\varepsilon_0\mu_0}\,\rho_m\right) = \frac{\rho_e'}{\varepsilon_0}$$

$$\mathrm{div}(\vec{B}') = \rho_m'\cdot\mu_0$$

$$\hookrightarrow \cos\underbrace{\mathrm{div}\vec{B}}_{\rho_m\mu_0} + \sin\sqrt{\varepsilon_0\mu_0}\underbrace{\mathrm{div}\vec{E}}_{\frac{\rho_e}{\varepsilon_0}}$$

$$= \mu_0\cdot\left(\cos\rho_m + \sin\frac{\rho_e}{\sqrt{\varepsilon_0\mu_0}}\right) = \mu_0\rho_m'$$

$$\mathrm{rot}(\vec{B}') = \mu_0\vec{j_e}' + \mu_0\varepsilon_0\frac{\partial\vec{E}'}{\partial t}$$

$$\hookrightarrow \underbrace{\mathrm{rot}(\vec{B})}_{\left(\mu_0\vec{j_e} + \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}\right)}\cos + \sqrt{\varepsilon_0\mu_0}\underbrace{\mathrm{rot}(\vec{E})}_{\left(-\mu_0\vec{j_m} - \frac{\partial\vec{B}}{\partial t}\right)}\sin$$

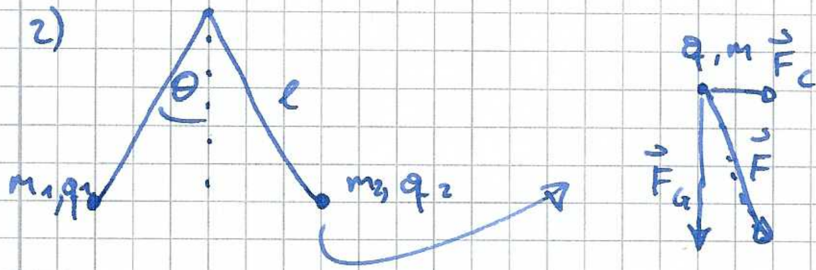
$$= \mu_0\cdot\underbrace{\left(\vec{j_e}\cos - \sqrt{\varepsilon_0\mu_0}\,\vec{j_m}\sin\right)}_{j_e'} + \underbrace{\left(\mu_0\varepsilon\frac{\partial\vec{E}}{\partial t}\cos - \sqrt{\mu_0\varepsilon_0}\frac{\partial\vec{B}}{\partial t}\sin\right)}_{\mu_0\varepsilon_0\left(\frac{\partial\vec{E}'}{\partial t}\right)}$$

$$\mathrm{rot}(\vec{E}') = -\mu_0 \vec{j_m}' - \frac{\partial \vec{B}'}{\partial t}$$

$$\underbrace{\mathrm{rot}(\vec{E})}_{\left(-\mu_0 \vec{j_m} - \frac{\partial \vec{B}}{\partial t}\right)} \cos - \underbrace{\mathrm{rot}(\vec{B})}_{\left(\mu_0 \vec{j_e} + \varepsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t}\right)} \sin \cdot \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$$

$$= -\mu_0 \Bigg(\underbrace{\vec{j_m} \cos + \frac{\vec{j_e}}{\sqrt{\varepsilon_0 \mu_0}} \sin}_{\vec{j_m}'} \Bigg) - \Bigg(\underbrace{\frac{\partial \vec{B}}{\partial t} \cos + \sqrt{\varepsilon_0 \mu_0} \frac{\partial \vec{E}}{\partial t}}_{+\frac{\partial \vec{B}'}{\partial t}} \Bigg)$$

2)



a) $|F_G| = mg \qquad |F_C| = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{(2l\sin\theta)^2} \quad , q_1 = q_2 = q$

$m_1 = m_2 = m$

$$\tan\theta = \frac{|F_C|}{|F_G|} = \frac{1}{4\pi\varepsilon_0 mg}\frac{q^2}{(2l\sin\theta)^2}$$

$$\tan\theta \simeq \sin\theta \simeq \theta$$

$$\rightarrow \theta = \sqrt[3]{\frac{1}{16\pi\varepsilon_0 mg}\left(\frac{q}{l}\right)^2}$$

b) $$q = \sqrt{\tan\theta\, \pi\varepsilon_0 mg}\cdot 4l\sin\theta$$

$$l = 0{,}15\,m \quad m = 10^{-3}\,kg \quad \theta = 7°$$

$$\rightarrow q = 1{,}92\cdot 10^{-7}\,C$$

c) $m_1 = \frac{m}{2}\ ,\ m_2 = m \rightarrow \frac{q_1}{q_2} = ?\ :\ \theta_1 \overset{!}{=} \theta_2$

$$\rightarrow \tan\theta_1 = \frac{1}{4\pi\varepsilon_0 mg}\frac{q_1 q_2}{(2l\sin\theta)^2}$$

$$\tan\theta_2 = \frac{1}{4\pi\varepsilon_0 mg}\frac{q_1 q_2}{(2l\sin\theta)^2}$$

↯

$$\rightarrow q_1 \overset{!}{=} 0 \vee q_2 \overset{!}{=} 0 \qquad \rightarrow \theta_1 = \theta_2 = 0$$

3) Vorbereitung: geladener Stab im Vakuum

$$\vec{\nabla}\cdot\vec{E} = 4\pi\rho$$

$$\rightarrow \int_V dV\, \vec{\nabla}\cdot\vec{E} = \int_V dV 4\pi\rho \qquad V\ldots \text{Zylinder}$$

$$\rightarrow \int_{\partial V} \vec{E}_r\cdot d\vec{A} = 4\pi q_{ein}$$

$$E_r r \int_0^{2\pi} d\varphi \int_0^h dz = 4\pi\tau\cdot h$$

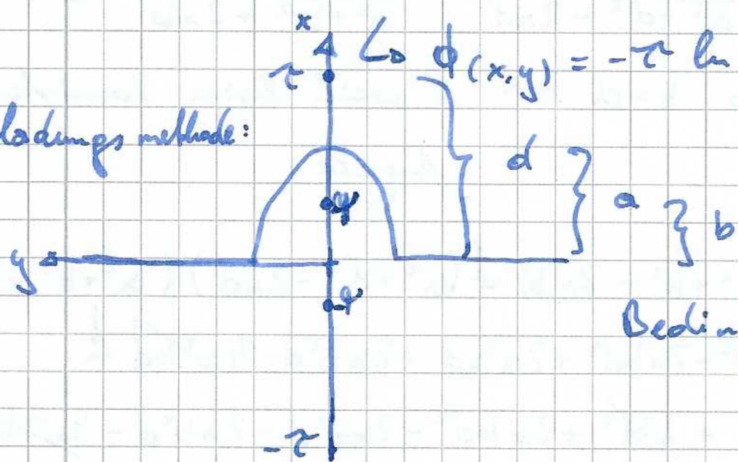
$$E_r 2r\cancel{\pi}\cancel{h} = \overset{2}{\cancel{4}}\cancel{\pi}\tau\cancel{h}$$

$$\rightarrow \vec{E} = \frac{2\tau}{r}\vec{e}_r = -\vec{\nabla}\phi$$

$$\rightarrow \phi = -2\tau\int dr\frac{1}{r} = -2\tau\ln(r) \quad \text{in Zylinderkoordinaten}$$

$$\hookrightarrow \phi_{(x,y)} = -\tau\ln(x^2+y^2)$$

Bildladungsmethode:



Bedingungen: $\phi_{(0,y)} \overset{!}{=} 0$

$$\phi_{(x,y)}\Big|_{x^2+y^2=a^2} \overset{!}{=} 0, \forall x>0$$

$$\lim_{x,y\to\infty}\phi_{(x,y)} \overset{!}{=} 0$$

$$\rightarrow \phi_{(x,y)} = \underbrace{-\tau\ln((x-d)^2+y^2)}_{\text{ursprüngliche Ladung}} + \underbrace{\tau\ln((x+d)^2+y^2)}_{\text{Ausgleich für 1. Bedingung}}$$

$$\underbrace{\mp\varphi\ln((x-b)^2+y^2)}_{\text{um 2. Bedingung zu rechtfertigen}} \pm \underbrace{\varphi\ln((x+b)^2+y^2)}_{\text{Ausgleich der Rechtfertigung der 2. Bed. für 1. Bed.}}$$

$$\to \phi_{(x,y)} = \ln\left(\frac{((x+d)^2+y^2)^{\tau}}{((x-d)^2+y^2)^{\tau}} \frac{((x+b)^2+y^2)^{\psi}}{((x-b)^2+y^2)^{\psi}}\right)$$

$$\to \phi_{(0,y)} = \ln\left(\left(\frac{d^2+y^2}{d^2+y^2}\right)^{\tau}\left(\frac{b^2+y^2}{b^2+y^2}\right)^{\psi}\right) = 0 \checkmark$$

$$\phi_{(x,y)}\Big|_{x^2+y^2=a^2} \stackrel{!}{=} 0$$

$$\to \phi_4 = \ln\left(\left(\frac{x^2+2xd+d^2+y^2}{x^2-2xd+d^2+y^2}\right)^{\tau}\left(\frac{x^2+2xb+b^2+y^2}{x^2-2xb+b^2+y^2}\right)^{\psi}\right) =$$

$$= \ln\left(\underbrace{\left(\frac{a^2+d^2+2xd}{a^2+d^2-2xd}\right)^{\tau}\left(\frac{a^2+b^2+2xb}{a^2+b^2-2xb}\right)^{\psi}}_{\stackrel{!}{=}1}\right)$$

$$\to \text{naiv}: \tau \stackrel{!}{=} -\psi$$

$$\to \frac{a^2+d^2+2xd}{a^2+d^2-2xd} \cdot \frac{a^2+b^2-2xb}{a^2+b^2+2xb} \stackrel{!}{=} 1$$

$\to b=d \to$ ↯ weil keine Konsistenz mit Angabe

$$\to (a^2+d^2+2xd)(a^2+b^2-2xb) \stackrel{!}{=} (a^2+d^2-2xd)(a^2+b^2+2xb)$$

$$\cancel{a^4}+\cancel{a^2b^2}-2xa^2b+\cancel{a^2d^2}+\cancel{b^2d^2}-2xbd^2+2a^2xd+2xb^2d-\cancel{4x^2bd} \stackrel{!}{=}$$

$$\cancel{a^4}+\cancel{a^2b^2}+2a^2xb+\cancel{a^2d^2}+\cancel{b^2d^2}+2xbd^2-2xa^2d-2xb^2d-\cancel{4x^2bd}$$

$$-\cancel{2x}((a^2+d^2)b-db^2-a^2d) \stackrel{!}{=} -\cancel{2x}(-b(a^2+d^2)+db^2+a^2d)$$

$$\to \cancel{2}db^2-\cancel{2}(a^2+d^2)b+\cancel{2}a^2d \stackrel{!}{=} 0$$

$$\to b_{1,2} = \frac{1}{2}\left(\frac{a^2}{d}+d\right) \pm \sqrt{\left(\frac{a^2}{d}+d\right)^2\frac{1}{4}-a^2}$$

$$= \frac{1}{2}\left(\frac{a^2}{d}+d\right) \pm \frac{1}{2}\sqrt{\frac{a^4}{d^2}+2a^2+d^2-4a^2} =$$

$$= \frac{1}{2}\left(\frac{a^2}{d}+d\right) \pm \frac{1}{2}\sqrt{\left(\frac{a^2}{d}-d\right)^2}$$

$$\to b_1 = d \ ↯ \qquad \underline{\underline{b_2 = \frac{a^2}{d}}}$$

$$\rightarrow \phi_{(x,y)} = \tau \ln\left(\frac{\left((x+d)^2+y^2\right)\left(\left(x-\frac{a^2}{d}\right)^2+y^2\right)}{\left((x-d)^2+y^2\right)\left(\left(x+\frac{a^2}{d}\right)^2+y^2\right)}\right)$$

$$\rightarrow \lim_{x,y\to\infty} \phi_{(x,y)} = 0 \quad \text{, da dann } x \gg \pm d; \frac{a^2}{d}$$

$\rightarrow$ alle Bedingungen erfüllt!

b) $\sigma = (4\pi)^{-1} E_n$

$$\rightarrow \sigma_{E(y)} = \frac{1}{4\pi} E_x \Big|_{x=0} = \frac{1}{4\pi}\left(-\frac{\partial \phi_{(x,y)}}{\partial x}\right)\Big|_{x=0} =$$

$$= \frac{-\tau}{4\pi}\left(\frac{2(x+d)}{(x+d)^2+y^2} + \frac{2\left(x-\frac{a^2}{d}\right)}{\left(x-\frac{a^2}{d}\right)^2+y^2} - \frac{2(x-d)}{(x-d)^2+y^2} - \frac{2\left(x+\frac{a^2}{d}\right)}{\left(x+\frac{a^2}{d}\right)^2+y^2}\right)\Bigg|_{x=0}$$

$$= \frac{-\tau}{2\pi}\left(\frac{d}{d^2+y^2} + \frac{-\frac{a^2}{d}}{\frac{a^4}{d^2}+y^2} - \frac{-d}{d^2+y^2} - \frac{\frac{a^2}{d}}{\frac{a^4}{d^2}+y^2}\right) = \underline{\underline{\frac{\tau d}{\pi}\left(\frac{a^2}{a^4+y^2d^2} - \frac{1}{d^2+y^2}\right)}}$$

$$\rightarrow \phi_{(x,y)} \rightarrow \phi_{(r,\varphi)} \; ; \; x = r\cos\varphi \;\&\; y = r\sin\varphi$$

$$\rightarrow \phi_{(r,\varphi)} = \tau \ln\left(\left(\frac{r^2+d^2+2dr\cos\varphi}{r^2+d^2-2dr\cos\varphi}\right)\left(\frac{r^2+\frac{a^4}{d^2}-2\frac{a^2}{d}r\cos\varphi}{r^2+\frac{a^4}{d^2}+2\frac{a^2}{d}r\cos\varphi}\right)\right)$$

$$\rightarrow \sigma_A = -\frac{1}{4\pi}\frac{\partial\phi}{\partial r}\Big|_{r=a} =$$

$$= \frac{-\tau}{4\pi}\left(\frac{2r+2d\cos\varphi}{r^2+d^2+2dr\cos\varphi} + \frac{2rd^2-2a^2d\cos\varphi}{r^2d^2+a^4-2a^2dr\cos\varphi}\right.$$
$$\left. - \frac{2r-2d\cos\varphi}{r^2+d^2-2dr\cos\varphi} - \frac{2rd^2+2a^2d\cos\varphi}{r^2d^2+a^4+2a^2dr\cos\varphi}\right)\Bigg|_{r=a}$$

$$= \frac{-\tau}{2\pi}\left(\frac{a+d\cos\varphi}{a^2+d^2+2da\cos\varphi} + \frac{ad^2-a^2d\cos\varphi}{a^2d^2+a^4-2a^3d\cos\varphi}\right.$$
$$\left. - \frac{a-d\cos\varphi}{a^2+d^2-2da\cos\varphi} - \frac{ad^2+a^2d\cos\varphi}{a^2d^2+a^4+2a^3d\cos\varphi}\right)$$

$c = \cos\varphi \Rightarrow$

$$= -\frac{\tau}{2\pi}\left(\frac{a+dc}{a^2+d^2+2adc} + \frac{ad^2-a^2dc}{a^2d^2+a^4-2a^3dc} - \frac{a-dc}{a^2+d^2-2dac} - \frac{ad^2+a^2dc}{a^2d^2+a^4+2a^3dc}\right)$$

$\to$ HN: a^2+d^2+2adc

$$\left.\begin{array}{l} a^2d^2+a^4-2a^3dc = a^2(a^2+d^2-2adc) \\ a^2+d^2-2adc \\ a^2d^2+a^4+2a^3dc = a^2(a^2+d^2+2adc) \end{array}\right\} a^2(a^2+d^2-2adc)(a^2+d^2+2adc)$$

$$\to = a^2\left(a^4+2a^2d^2+d^4-4a^2d^2c^2\right)$$

$$\left(= 2a^2d^2(c^2+s^2)\right) \qquad , s \hat{=} \sin\varphi$$

$$= a^2\left(a^4+d^4 + \underbrace{2a^2d^2s^2 - 2a^2d^2c^2}_{= -2a^2d^2\cos(2\varphi)}\right)$$

$\to$ HN: $a^2(a^4+d^4-2a^2d^2\cos(2\varphi))$

$\to$ Zähler: $(a+dc)a^2(a^2+d^2-2adc)+(ad^2-a^2dc)(a^2+d^2+2adc)$
$-(a-dc)(a^2+d^2+2adc)a^2-(ad^2+a^2dc)(a^2+d^2-2adc) =$

$= \cancel{a^5}+\cancel{a^3d^2}-2a^4dc+\cancel{a^4dc}+\cancel{a^2d^3c}-\cancel{2a^3d^2c^2}+\cancel{a^3d^2}+\cancel{ad^4}+2a^2d^3c-\cancel{a^4dc}-\cancel{a^2d^3c}$
$-\cancel{2a^3d^2c^2}-\cancel{a^5}-\cancel{a^3d^2}-2a^4dc+\cancel{a^4dc}+\cancel{a^2d^3c}+\cancel{2a^3d^2c^2}-\cancel{a^3d^2}-\cancel{ad^4}+2a^2d^3c$
$-\cancel{a^4dc}-\cancel{a^2d^3c}+\cancel{2a^3d^2c^2} =$

$= -4a^4dc+4a^2d^3c = 4a^2dc(d^2-a^2)$

$$\to \sigma_A = \frac{-\tau}{2\pi}\frac{4a^2dc(d^2-a^2)}{a^2(a^4+d^4-2a^2d^2\cos(2\varphi))} = \boxed{\frac{2\tau(a^2-d^2)d\cos(\varphi)}{\pi(a^4+d^4-2a^2d^2\cos(2\varphi))}}$$

$\to q = \int \sigma \, dl$

$$\to q_E = 2\int_a^\infty \sigma_E\,dy = \frac{2\tau d}{\pi}\int_a^\infty dy\left(\frac{a^2}{a^4+y^2d^2}-\frac{1}{d^2+y^2}\right) =$$

$$= \frac{2\tau d}{\pi}\int_a^\infty dy\left(\frac{a^2}{a^4}\frac{1}{1+\frac{y^2d^2}{a^4}}-\frac{1}{d^2}\frac{1}{1+\frac{y^2}{d^2}}\right) = \qquad \frac{yd}{a^2}=\alpha \to dy\frac{d}{a^2}=d\alpha$$

$$\frac{y}{d}=\beta \to dy\frac{1}{d}=d\beta$$

$$= \frac{2\tau d}{\pi}\left(\frac{1}{\cancel{a^2}}\cancel{\arctan\left(\frac{yd}{a^2}\right)}\int d\alpha\frac{\cancel{a^2}}{d}\frac{1}{1+\alpha^2}-\frac{1}{d^2}\int d\beta\, d\frac{1}{1+\beta^2}\right) =$$

$$= \frac{2\tau \cancel{d}}{\pi}\left(\frac{1}{\cancel{d}}\arctan\left(\frac{yd}{a^2}\right)-\frac{1}{\cancel{d}}\arctan\left(\frac{y}{d}\right)\right)\Bigg|_{y=a}^{\infty} =$$

$$\frac{2\tau}{\pi}\left(\cancel{\frac{\pi}{2}} - \arctan\left(\frac{d}{a}\right) - \cancel{\frac{\pi}{2}} + \arctan\left(\frac{a}{d}\right)\right) =$$

$$= \frac{2\tau}{\pi}\left(\arctan\left(\frac{\frac{a}{d} - \frac{d}{a}}{1 + \frac{a}{d}\frac{d}{a}}\right)\right) = \underline{\underline{\frac{2\tau}{\pi}\arctan\left(\frac{a^2-d^2}{2ad}\right)}}$$

$$\rightarrow q_A = 2\int_0^{\frac{\pi}{2}} d\varphi\, r\, \sigma_A \Big|_{r=a} =$$

$$= \frac{4\tau}{\pi} ad(a^2-d^2)\int_0^{\frac{\pi}{2}} d\varphi \frac{\cos(\varphi)}{\underbrace{a^4+d^4-2a^2d^2\cos(2\varphi)}}$$

$$= a^4 + 2a^2d^2 + d^4 - 4a^2d^2c^2$$

$$= a^4 + 2a^2d^2 + d^4 - 4a^2d^2 + 4a^2d^2s^2$$

$$= (a^2-d^2)^2 + 4a^2d^2s^2$$

$$\gamma = \sin(\varphi) \rightarrow d\gamma = \cos(\varphi)\, d\varphi$$

$$= \frac{4\tau}{\pi} ad(a^2\cancel{-d^2})\int d\gamma \frac{\cancel{\cos\varphi}}{\cancel{\cos\varphi}} \frac{1}{\cancel{(a^2-d^2)^2}\left(1 + \frac{4a^2d^2\gamma^2}{(a^2-d^2)^2}\right)} =$$

$$= \frac{4\tau ad}{\pi(a^2-d^2)}\int d\gamma \frac{1}{1 + \frac{4a^2d^2\gamma^2}{(a^2-d^2)^2}}$$

$$\frac{2ad}{(a^2-d^2)}\gamma = \xi$$

$$\rightarrow \frac{2ad}{(a^2-d^2)} d\gamma = d\xi$$

$$= \frac{\overset{2}{\cancel{4}}\tau \cancel{ad}}{\pi\cancel{(a^2-d^2)}} \frac{\cancel{(a^2-d^2)}}{\cancel{2ad}}\int d\xi \frac{1}{1+\xi^2} =$$

$$= \frac{2\tau}{\pi} \arctan\left(\frac{2ad}{(a^2-d^2)}\gamma\right)\Big|_{\gamma=\sin\varphi} =$$

$$= \frac{2\tau}{\pi}\arctan\left(\frac{2ad}{(a^2-d^2)}\sin\varphi\right)\Big|_{\varphi=0}^{\frac{\pi}{2}} =$$

$$= \frac{2\tau}{\pi}\left(\arctan\left(\frac{2ad}{(a^2-d^2)}\right) - 0\right) = \frac{2\tau}{\pi}\arctan\left(\frac{2ad}{a^2-d^2}\right)$$

$$\rightarrow q_{ges} = q_E + q_A = \frac{2\tau}{\pi}\left(\arctan\left(\frac{a^2-d^2}{2ad}\right) + \arctan\left(\frac{2ad}{(a^2-d^2)}\right)\right) =$$

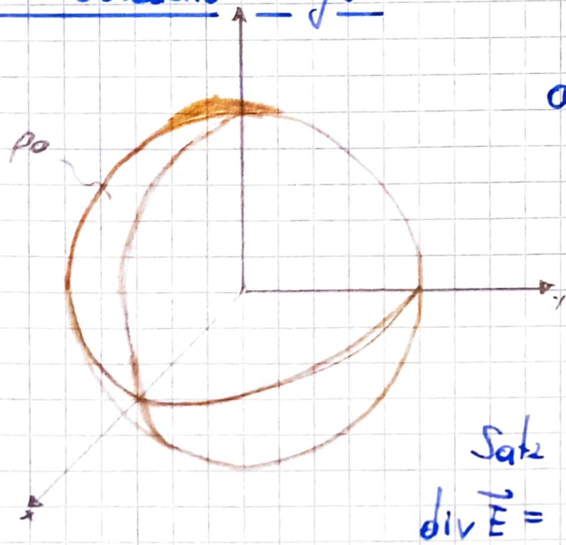
$$= \frac{2\tau}{\pi}\left(\arctan\left(\frac{\frac{a^2-d^2}{2ad} + \frac{2ad}{a^2-d^2}}{1 - \frac{(a^2-d^2)}{2ad}\frac{2ad}{(a^2-d^2)}}\right) - \pi\right)$$

$$= \frac{2\tau}{\pi}\left(\arctan\left(\frac{\frac{a^2-d^2}{2ad} + \frac{2ad}{a^2-d^2}}{0}\right) - \pi\right) =$$

$$= \frac{2\tau}{\pi}\left(\arctan\left(\infty\right) - \pi\right) = \cancel{\arctan}\ \frac{2\tau}{\pi}\left(\frac{\pi}{2} - \pi\right) =$$

$$= \frac{2\tau}{\pi}\left(-\frac{\pi}{2}\right) = \underline{\underline{-\tau}}$$

5.4 Geladene Kugel



a) Probe volle Kugel

→ Symmetrie

Richtung $\vec{e}_r$

$\rho = \rho_0 \cdot H(R-r)$

homogen geladen

Satz von Gauss

$\operatorname{div}\vec{E} = 4\pi\rho \quad / \int$

$$\underbrace{\int \operatorname{div}\vec{E}\, dV}_{\int \vec{E}\, d\vec{A}} = 4\pi \int \rho\, dV$$

$\vec{n} = \vec{e}_r$

außerhalb

$Q = \int \rho\, dV = \frac{4\pi}{3} R^3 \rho_0$

$$E \cdot \int r^2 \sin\theta\, d\theta\, d\varphi = 4\pi \frac{4\pi}{3} R^3 \rho_0$$

$$E(r) = \frac{4\pi R^3 \rho_0}{3 r^2} = \frac{Q}{r^2}$$

innerhalb

$$E \cdot r^2 \cdot 4\pi = 4\pi \cdot \frac{4\pi}{3} r^3 \cdot \rho_0$$

$$E(r) = \frac{4\pi}{3} r \cdot \rho_0 = \frac{Q}{R^3} \cdot r$$

$$\vec{E}(r) = \left(\frac{4\pi}{3} r \rho_0 \cdot H(R-r) + \frac{4\pi}{3} \frac{R^3 \rho_0}{r^2} H(r-R)\right) \cdot \vec{e}_r$$

$$= \left(\frac{Q}{r^2} H(r-R) + \frac{Q}{R^3} \cdot r\, H(R-r)\right) \cdot \vec{e}_r$$