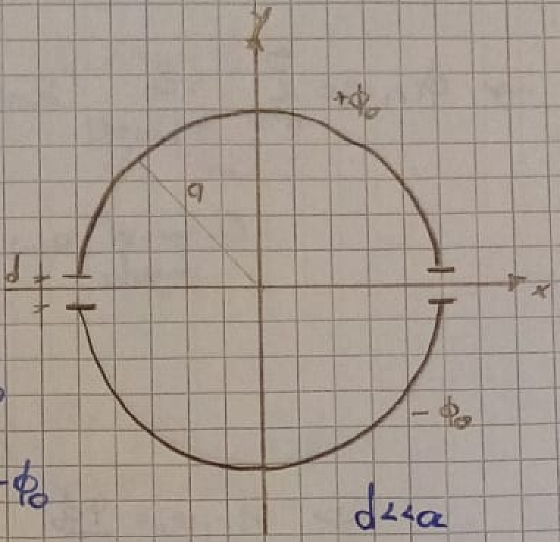
8.1 Geteilter Zylinder

a) Potential Innen $r < a$

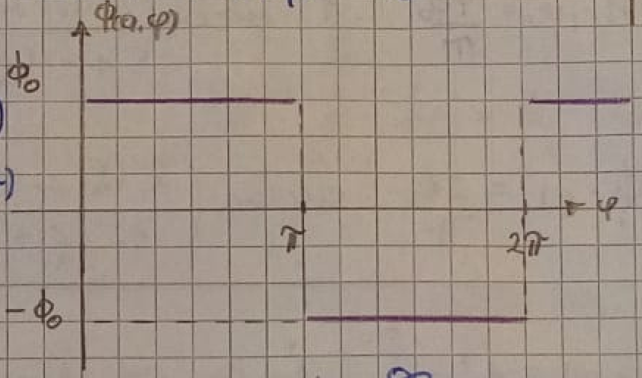


Ansatz: $\Phi(r,\varphi) = A_0 + \sum_{m=1}^{\infty} [A_m \cos(m\varphi) + B_m \sin(m\varphi)] \left(\frac{r}{a}\right)^m$

für $\varphi \in (0, \pi) \rightarrow \Phi(a,\varphi) = +\Phi_0$

$\varphi \in (\pi, 2\pi) \rightarrow \Phi(a,\varphi) = -\Phi_0$

$d \ll a$



$$f(\varphi) = \begin{cases} +\Phi_0 & (0,\pi) \\ -\Phi_0 & [\pi, 2\pi) \end{cases}$$

$\rightarrow$ ungerade Funktion

$\Rightarrow$ für Entw. $A_m = 0$

$$\Phi(a,\varphi) = A_0 + \sum_{m=1}^{\infty} B_m \sin(m\varphi) \cdot \left(\frac{a}{a}\right)^m$$

$$= \underbrace{A_0}_{0} + \sum_{m=1}^{\infty} B_m \sin(m\varphi) = f(\varphi)$$

$\}$ 0, weil Konstante u. für $r = 0$ soll $\Phi = 0$ gelten! (gewählt)

$\sum_{m=1}^{\infty} B_m \cdot \sin(m\varphi) = f(\varphi)$ $\rightarrow$ vgl. Fourierentwicklung

$$\rightarrow B_m = \frac{1}{\pi} \int_0^{2\pi} f(\varphi) \cdot \sin(m\varphi)\, d\varphi = \frac{1}{\pi}\left(\int_0^{\pi} \Phi_0 \sin(m\varphi)\, d\varphi - \int_{\pi}^{2\pi} \Phi_0 \sin(m\varphi)\, d\varphi \right)$$

$$= \frac{\Phi_0}{\pi}\left(-\frac{\cos(m\varphi)}{m}\Big|_0^{\pi} + \frac{\cos(m\varphi)}{m}\Big|_{\pi}^{2\pi} \right) = \frac{\Phi_0}{\pi \cdot m}\left(-(-1)^m + 1 + 1 - (-1)^m\right)$$

$$B_m = \frac{2\Phi_0}{\pi \cdot m} \cdot (1 - (-1)^m)$$

$$\Phi(r,\varphi) = \sum_{m=1}^{\infty} \frac{2\Phi_0}{\pi \cdot m} (1 - (-1)^m) \sin(m\varphi) \left(\frac{r}{a}\right)^m$$

$\rightarrow 0$ für $m = 2 \cdot n$ $n \in \mathbb{N}$, Transf. $m = 2n+1$

$$\Phi(r,\varphi) = \sum_{n=0}^{\infty} \frac{2\Phi_0}{\pi(2n+1)} \cdot 2 \cdot \sin((2n+1)\cdot\varphi) \left(\frac{r}{a}\right)^{2n+1}$$

$\rightarrow \phi(r,\varphi) = \sum_{n=0}^{\infty} \frac{4\phi_0}{\pi(2n+1)} \sin((2n+1)\varphi) \left(\frac{r}{a}\right)^{2n+1}$ $\quad a > r$

el. Formel aus Angabe $\quad \sum_{n=0}^{\infty} p^{2n+1} \frac{\sin((2n+1)\varphi)}{2n+1} = \frac{1}{2} \arctan\left(\frac{2p \sin\varphi}{1-p^2}\right)$

$0 < \varphi < 2\pi, \; p^2 \leq 1$

$$\Rightarrow \phi(r,\varphi) = \frac{4\phi_0}{\pi} \cdot \frac{1}{2} \arctan\left(\frac{2 \cdot \left(\frac{r}{a}\right) \sin(\varphi)}{1 - \left(\frac{r}{a}\right)^2}\right)$$

$$\phi(r,\varphi) = \frac{2\phi_0}{\pi} \cdot \arctan\left(\frac{2\left(\frac{r}{a}\right)\sin\varphi}{1-\left(\frac{r}{a}\right)^2}\right) \quad \text{für } r < a$$

b) Ansatz: $\phi(r,\varphi) = A_0' + \sum_{m=1}^{\infty} \left[A_m' \cos(m\varphi) + B_m' \sin(m\varphi)\right] \left(\frac{a}{r}\right)^m$ $\quad r > a$

analoge Rechnung zu a) $\quad A_0' = \emptyset$ weil Potential stetig an GF u. entsprechend von Wahl A_0 in a)

$$\Rightarrow \phi_a(r,\varphi) = \frac{2\phi_0}{\pi} \arctan\left(\frac{2\left(\frac{a}{r}\right)\sin\varphi}{1-\left(\frac{a}{r}\right)^2}\right) \quad \text{für } r > a$$

c) Flächenladungsdichte

$$\sigma_i = -\frac{1}{4\pi} \left.\frac{\partial \phi(r,\varphi)}{\partial r}\right|_{r=a}$$

$$\left.\frac{\partial \phi(r,\varphi)}{\partial r}\right|_{r=a} = \frac{2\phi_0}{\pi} \cdot \frac{1}{\left(\frac{2\left(\frac{r}{a}\right)\sin\varphi}{1-\left(\frac{r}{a}\right)^2}\right)^2 + 1} \cdot \frac{\frac{2}{a}\sin\varphi \cdot \left(1-\left(\frac{r}{a}\right)^2\right) - \frac{2r}{a}\sin\varphi \cdot \left(-\left(\frac{2r}{a^2}\right)\right)}{\left(1-\left(\frac{r}{a}\right)^2\right)^2}$$

$$= \frac{2\phi_0}{\pi} \cdot \left.\frac{2\sin\varphi \cdot \left(\frac{1}{a}\left(1-\left(\frac{r}{a}\right)^2\right) + \frac{2r^2}{a^3}\right)}{\left(2\left(\frac{r}{a}\right)\sin\varphi\right)^2 + \left(1-\left(\frac{r}{a}\right)^2\right)^2}\right|_{r=a} = \frac{4\phi_0}{\pi} \sin\varphi \frac{2 \cdot \frac{1}{a}}{4\sin\varphi^2} = \frac{2\phi_0}{\pi \sin\varphi \, a}$$

$$\sigma(\varphi) = \frac{2\phi_0}{\pi a \cdot \sin\varphi} \cdot \frac{1}{4\pi} = \frac{\phi_0}{2\pi^2 \cdot a \sin\varphi} \quad \text{(innen)}$$

→ analoges Ergebnis für äußeres ϕ

$$\Rightarrow \quad \underline{\sigma(\varphi) = \frac{\phi_0}{\pi^2 a \sin\varphi}} \qquad (\vec{\sigma}_i + \vec{\sigma}_a)$$

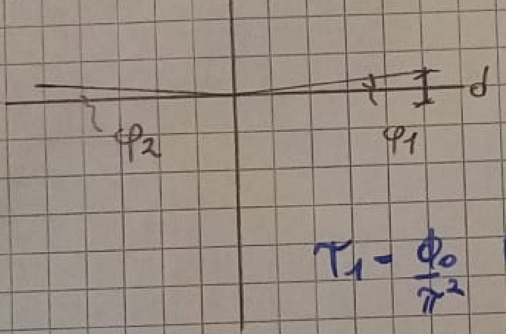
d)

$$Q = \int \sigma \, \underbrace{dA}_{a \cdot d\varphi\, dz} = \underbrace{\int dz}_{L} \; a \cdot \int \left(\frac{\phi_0}{\pi^2 a \sin\varphi} \right) d\varphi$$

$\tau = \frac{Q}{L}$

$$\tau_1 = \frac{\phi_0}{\pi^2} \int \frac{1}{\sin\varphi} \, d\varphi = \frac{\phi_0}{\pi^2} \left(\log\left|\tan\left(\frac{\varphi}{2}\right)\right| \right) \Big|_{\varphi_1}^{\varphi_2}$$

$\varphi \in (0, \pi)$



$$\varphi_1 = \tan\left(\frac{d}{2a}\right) \sim \frac{d}{2a}$$

$$\varphi_2 \mathrel{\hat{=}} \pi - \frac{d}{2a}$$

$$\tau_1 = \frac{\phi_0}{\pi^2} \log\left(\frac{\tan\left(\frac{\pi}{2} - \frac{d}{4a}\right)}{\tan\left(\frac{d}{4a}\right)} \right)$$

$$= \frac{\phi_0}{\pi^2} \log\left(\frac{\cot\left(\frac{d}{4a}\right)}{\tan\left(\frac{d}{4a}\right)} \right) = \frac{\phi_0}{\pi^2} \log\left(\frac{1}{\tan\left(\frac{d}{4a}\right)^2} \right)$$

$$\underline{\tau_1 = \frac{2\phi_0}{\pi^2} \cdot \log\left(\frac{4a}{d} \right)} \qquad \text{für } \varphi \in (0, \pi)$$

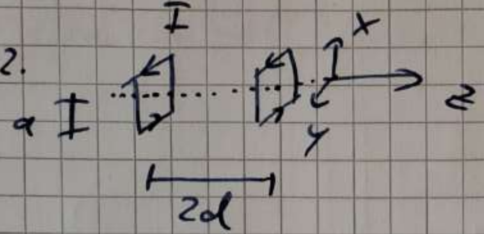
→ analog

$$\underline{\tau_2 = \frac{-2\phi_0}{\pi^2} \log\left(\frac{4a}{d} \right)} \qquad \text{für } \varphi \in (\pi, 2\pi)$$

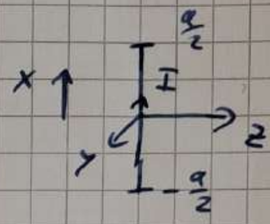
Kapazität $C = \frac{Q}{U}$ $\qquad c = \frac{\tau}{U}$ (pro Längeneinheit)

$$\underline{c = \frac{\tau_1}{2\phi_0} = \frac{1}{\pi^2} \log\left(\frac{4a}{d} \right)}$$

8.2.



a) Zuerst: 1 Leiter in x-Richtung bei $y=z=0$



$\Rightarrow \vec{j} = I\cdot\theta\left(\frac{a}{2}+x\right)\theta\left(\frac{a}{2}-x\right)\delta(y)\,\delta(z)\,\hat{e}_x$

Biot-Savart: $\vec{B} = \frac{1}{c}\int dV' \, \frac{\vec{j}(\vec{r}')\times(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}$ $\quad \vec{r}' = \begin{pmatrix} x' \\ 0 \\ 0 \end{pmatrix}$

$$\Rightarrow \vec{B}_{1x} = \frac{I}{c}\int_{-\frac{a}{2}}^{\frac{a}{2}} dx' \, \frac{\begin{pmatrix}1\\0\\0\end{pmatrix}\times\begin{pmatrix}x-x'\\y\\z\end{pmatrix}}{\left((x-x')^2+y^2+z^2\right)^{3/2}} = \left| \begin{array}{l} x-x'=u \\ -dx'=du \end{array}\right|$$

$$= \frac{I}{c}\int_{x-\frac{a}{2}}^{x+\frac{a}{2}} du \, \frac{\begin{pmatrix}0\\-z\\y\end{pmatrix}}{\left(u^2+y^2+z^2\right)^{3/2}} = \left|\text{Hint.}\right| = \frac{I}{c}\begin{pmatrix}0\\-z\\y\end{pmatrix}\frac{u}{(y^2+z^2)\sqrt{u^2+y^2+z^2}}\Bigg|_{x-\frac{a}{2}}^{x+\frac{a}{2}}$$

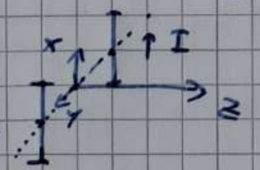
$$= \frac{I}{c}\begin{pmatrix}0\\-z\\y\end{pmatrix}\left[\frac{x+\frac{a}{2}}{(y^2+z^2)\sqrt{\left(x+\frac{a}{2}\right)^2+y^2+z^2}} - \frac{x-\frac{a}{2}}{(y^2+z^2)\sqrt{\left(x-\frac{a}{2}\right)^2+y^2+z^2}}\right]$$

Wir sind nur an z-achse interessiert! $\Rightarrow x = 0$

$$\vec{B}_{1x} = \frac{Ia}{c}\begin{pmatrix}0\\-z\\y\end{pmatrix}\frac{1}{y^2+z^2}\,\frac{1}{\sqrt{\left(\frac{a}{2}\right)^2+y^2+z^2}}$$

Verschiebung zu $y=-\frac{a}{2} \rightarrow I$, $\quad y=+\frac{a}{2} \rightarrow -I$ (Richtung!)

$\rightarrow$ 2 Stäbe:
(Paar)



$$\vec{B}_{2x} = \vec{B}_{1x}\left(y+\frac{a}{2}, z\right) - \vec{B}_{1x}\left(y-\frac{a}{2}, z\right) \overset{\overset{\text{nur z-Achse!}}{y=0}}{=}$$

$$= \frac{Ia}{c}\left[\begin{pmatrix}0\\-z\\a/2\end{pmatrix}\frac{1}{\frac{a^2}{4}+z^2}\,\frac{1}{\sqrt{\left(\frac{a}{2}\right)^2+\left(\frac{a}{2}\right)^2+z^2}} - \begin{pmatrix}0\\-z\\-a/2\end{pmatrix}\frac{1}{a^2/4+z^2}\,\frac{1}{\sqrt{\left(\frac{a}{2}\right)^2+\left(\frac{a}{2}\right)^2+z^2}}\right]$$

$$= \frac{Ia^2}{c}\begin{pmatrix}0\\0\\1\end{pmatrix}\frac{1}{\left(\frac{a}{2}\right)^2+z^2}\,\frac{1}{\sqrt{\frac{a^2}{2}+z^2}}$$

$\Rightarrow$ 1 Spule bei $z=0$: $\vec{B}_1 = 2\cdot \vec{B}_{2x}$ $\rightarrow$ egal, ob Paar um z-gedreht wurde!

$$= \frac{2Ia^2}{c}\hat{e}_z \frac{1}{\left(\frac{a}{2}\right)^2+z^2}\frac{1}{\sqrt{\frac{a^2}{2}+z^2}}$$

$\Rightarrow$ 2 Spulen bei $z=\pm d$

$$\Rightarrow \vec{B} = \vec{B}_1(z-d) + \vec{B}_1(z+d) =$$

$$= \frac{2Ia^2}{c}\hat{e}_z \left[\frac{1}{\left(\frac{a}{2}\right)^2+(z-d)^2}\cdot\frac{1}{\sqrt{\frac{a^2}{2}+(z-d)^2}} + \frac{1}{\left(\frac{a}{2}\right)^2+(z+d)^2}\cdot\frac{1}{\sqrt{\frac{a^2}{2}+(z+d)^2}}\right]$$

b) $\frac{\partial \vec{B}}{\partial z} = \cancel{0} \overset{?}{=} \overset{!}{=} \emptyset$

$$\frac{\partial \vec{B}_1}{\partial z} = \frac{2Ia^2}{c}\hat{e}_z\left[\frac{-2z}{\left(\left(\frac{a}{2}\right)^2+z^2\right)^2}\frac{1}{\sqrt{\frac{a^2}{2}+z^2}} + \frac{1}{\left(\frac{a^2}{2}+z^2\right)}\cdot\frac{-\frac{1}{2}\cdot 2z}{\left(\frac{a^2}{2}+z^2\right)^{3/2}}\right]$$

$$= \frac{2Ia^2}{c}\hat{e}_z\frac{-z}{\left(\left(\frac{a}{2}\right)^2+z^2\right)\sqrt{\frac{a^2}{2}+z^2}}\left[\frac{2}{\left(\frac{a}{2}\right)^2+z^2}+\frac{1}{\frac{a^2}{2}+z^2}\right]$$

$$\Rightarrow \frac{\partial \vec{B}}{\partial z} \propto \hat{e}_z \times \left[\frac{-(z-d)}{\left(\left(\frac{a}{2}\right)^2+(z-d)^2\right)\sqrt{\frac{a^2}{2}+(z-d)^2}}\left(\frac{2}{\left(\frac{a}{2}\right)^2+(z-d)^2}+\frac{1}{\frac{a^2}{2}+(z-d)^2}\right) + \frac{-(z+d)}{\left(\left(\frac{a}{2}\right)^2+(z+d)^2\right)\sqrt{\frac{a^2}{2}+(z+d)^2}}\underbrace{\left(\frac{2}{\left(\frac{a}{2}\right)^2+(z+d)^2}+\frac{1}{\frac{a^2}{2}+(z+d)^2}\right)}_{\frac{a^2+(z\pm d)^2+\left(\frac{a}{2}\right)^2+(z\pm d)^2}{\left(\left(\frac{a}{2}\right)^2+(z\pm d)^2\right)\left(\frac{a^2}{2}+(z\pm d)^2\right)}}\right]$$

$z=\emptyset \Rightarrow \frac{\partial \vec{B}}{\partial z} = \emptyset$

$$\frac{\partial B_z}{\partial z} \propto (z-d)\left(\left(\frac{a}{2}\right)^2+(z+d)^2\right)\sqrt{\frac{a^2}{2}+(z+d)^2}\left(\frac{\frac{5a^2}{4}+(z-d)^2}{\left(\left(\frac{a}{2}\right)^2+(z-d)^2\right)\left(\frac{a^2}{2}+(z-d)^2\right)}\right)$$

$$+\ (z\mp d \to z\pm d)$$

$$\propto (z-d)\left(\left(\frac{a}{2}\right)^2+(z+d)^2\right)\sqrt{\frac{a^2}{2}+(z+d)^2}\left(\frac{5a^2}{4}+(z-d)^2\right)\times$$

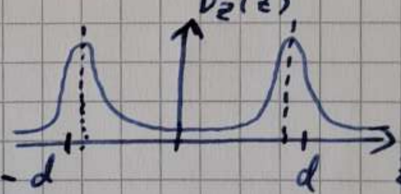
$$\times\left(\left(\left(\frac{a}{2}\right)^2+(z+d)^2\right)\left(\frac{a^2}{2}+(z+d)^2\right)\right)$$

$$+\ (z\mp d \to z\pm d)$$

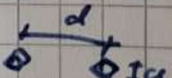
$$\propto (z-d)\left(\left(\frac{a}{2}\right)^2+(z+d)^2\right)^2\left(\frac{a^2}{2}+(z+d)^2\right)^{3/2}\left(\frac{5a^2}{4}+(z-d)^2\right)$$

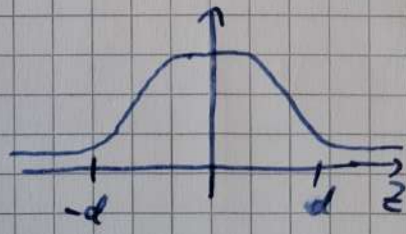
$$+\ (z\mp d \to z\pm d)$$

solvable for $a \approx 4.2d$ $\Rightarrow$

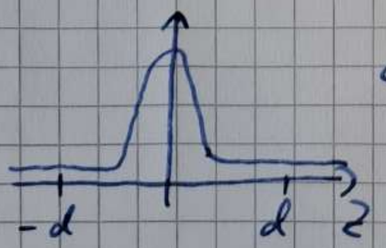


$a \ll 4.2d$





$a \approx 4.2d$ (Helmholtz-Paar)



$a \gg 4.2d$ (sehr eng)