3. Test

$S \longleftrightarrow T$

1 (a) Potentiale:

$V \updownarrow p$

$E(S,V,N)$ ✓ +0,5P $\qquad F(T,V,N)$ ✓ +0,5P

$H(S,p,N)$ ✓ +0,5P $\qquad G(T,p,N)$ ✓ +0,5P

$F(T,V,N) = E(S,V,N) - ST$ ✓ +0,5P mit S sodass $\left.\frac{\partial E}{\partial S}\right|_{V,N} = T$ ✓ +0,5P

$H(S,p,N) = E(S,V,N) + pV$ ✓ +0,5P mit V sodass $\left.\frac{\partial E}{\partial V}\right|_{S,N} = -p$ ✓ +0,5P

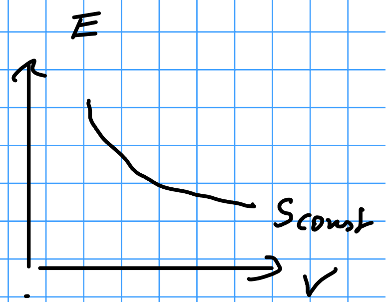
$G(T,p,N) = H(S,p,N) - ST$ ✓ +0,5P mit S sodass $\left.\frac{\partial H}{\partial S}\right|_{p,N} = T$ ✓ +0,5P

oder $G(T,p,N) = F(T,V,N) + pV$ ✓ +0,5P mit V sodass $\left.\frac{\partial F}{\partial V}\right|_{T,N} = -p$ ✓ +0,5P

(b) kalorische & thermische Zustandsgleichung:

(I) $$E = \theta \frac{N^3 (k_B T)^2}{V^2}$$

(II) $$p = \frac{2E}{V}$$



$$f(x,y): \left.\frac{\partial f}{\partial x}\right|_g = \left.\frac{\partial f}{\partial x}\right|_y + \left.\frac{\partial f}{\partial y}\right|_x \underbrace{\left.\frac{\partial y}{\partial x}\right|_g}_{= -\left(\frac{\partial g}{\partial x}\right)\left(\frac{\partial g}{\partial y}\right)^{-1}} \quad (*)$$

damit ($f = p$, $x = V$, $y = E$, $g = S$)

$$\left.\frac{\partial p}{\partial V}\right|_{S,N} = \underbrace{\left.\frac{\partial p}{\partial V}\right|_E}_{= -\frac{2E}{V^2}} + \underbrace{\left.\frac{\partial p}{\partial E}\right|_V}_{= \frac{2}{V}} \underbrace{\left.\frac{\partial E}{\partial V}\right|_S}_{= -p = -\frac{2E}{V}} \overset{(II)}{=} -\frac{2E}{V^2} - \frac{4E}{V^2} = \underline{-\frac{6E}{V^2}}$$

(*) angewandt: +1P ✓ +1P ✓ +1P ✓ +1P

$$\Rightarrow \underline{K_S} = -\frac{1}{V} \underbrace{\left.\frac{\partial V}{\partial p}\right|_{S,N}}_{= \left(\frac{\partial p}{\partial V}\right)^{-1}} = -\frac{1}{V}\left(-\frac{6E}{V^2}\right)^{-1} = \frac{V^{\cancel{2}}}{\cancel{V}6E} \overset{(I)}{=} \underline{\underline{\frac{V^3}{6\theta N^3 (k_B T)^2}}}$$

$= \left(\frac{\partial p}{\partial V}\right)^{-1}$: +1P

✓ +1P

2 Dipole mit diskreten Ausrichtungen

$$H(\underline{q}) = -m\sum_{n=1}^{N} \vec{q}_n\cdot\vec{B} = -mB\sum_n \vec{q}_n\cdot\hat{z} \quad \text{mit } \vec{q}_n\cdot\hat{z}\in\{+1,-1\}$$

(a) k seinen // zu $+\hat{z}$, d.h. $\vec{q}_n\cdot\hat{z}=+1$ ✓ k mal oBdA $q_1\ldots q_k$

+1P: $\vec{q}_n\cdot\hat{z}=+1$ k mal

$\vec{q}_n\cdot\hat{z}=-1$ ✓ +1P $N-k$ mal $q_{k+1}\ldots q_N$

$$\Rightarrow \underline{E_k} = -mB\left[\sum_{n=1}^{k}(+1) + \sum_{n=k+1}^{N}(-1)\right] = -mB\left[k - (N-k)\right] = -mB(N-2k)$$

✓ ✓ ✓ +1P $\sum_{n=1}^{k}(+1)=k$ ✓ +1P

$$= \underline{mB(2k-N)} \Rightarrow k = \left(\frac{E_k}{mB}+N\right)/2, \quad E_k\in[-mBN,+mBN]$$

$$\#\text{ Mikrozustände} = \binom{N}{k} \quad ✓ +2P \quad \Rightarrow \underline{\Omega(E_k,N) = \binom{N}{\left(\frac{E_k}{mB}+N\right)/2}}$$

$$\left(S(E_k,N) = k_B\log\Omega(E_k,N) \approx k_B\log(N!) - k_B\log\{[(E_k/mB+N)/2]!\} - k_B\log\{[(-E_k/mB+N)/2]!\} \right)$$

+1P $S(E,N)$ als Fu. von E

(b)
$$\underline{Z_k} = \sum_k \Omega(E_k,N) = \sum_{k=0}^{N}\binom{N}{k}e^{-\beta E_k} = \sum_{k=0}^{N}\binom{N}{k}e^{\beta mBN}\left(e^{-2\beta mB}\right)^k$$

✓ +1P   +1P: E_k einsetzen

$$= e^{\beta mBN}\sum_{k=0}^{N}\binom{N}{k}\left(e^{-2\beta mB}\right)^k\cdot(1)^{N-k}$$

+1P: konst. Term ✓ ✓+2P auf Binomial Form bringen +2P Binomialsatz anwenden

$$= e^{\beta mBN}\left(e^{-2\beta mB}+1\right)^N = \left[e^{\beta mB}\left(e^{-2\beta mB}+1\right)\right]^N$$

$$= \left[e^{-\beta mB}+e^{+\beta mB}\right]^N = \underline{\left(2\cosh\beta mB\right)^N}$$

✓+1P symmetrische Form für cosh

(c) Seien nun $1\ll k,N$. Mit $\log(N!)\approx N\log N$
und $k(E) = (E/mB+N)/2$

+1P: Abschätzung für $\log\Omega$

$$S(E,N)\approx k_B\left[N\log N - k(E)\log(k(E)) - (N-k(E))\log(N-k(E))\right]$$

$$T = \left(\frac{\partial S}{\partial E}\Big|_N\right)^{-1} = \left[k_B\underbrace{\left(-\log(k(E)) - \cancel{1} + \log(N-k(E)) + \cancel{1}\right)}_{<0}\underbrace{\frac{\partial k(E)}{\partial E}}_{=\frac{1}{2mB}>0}\right]^{-1}$$

✓+1P ✓+0,5P

$\Rightarrow T$ negativ, wenn <0

d.h. $\log(N-k) < \log(k) \overset{\log\text{ monoton}}{\Longleftrightarrow} N-k<k \Leftrightarrow \underline{k>N/2}$ Besetzungsinversion

✓+0,5P

3 offenes System von Fermionen mit 2 Plätzen

$\mathcal{B} = (|00\rangle, |10\rangle, |01\rangle, |11\rangle)$

$\hat{=}$ # Fermionen auf 2. Platz

(a) $\langle 10|\hat{H}|01\rangle = t/2 \xRightarrow[\text{tauschen}]{\text{Plätze}} \langle 01|\hat{H}|10\rangle = t/2$

$\langle 10|\hat{H}|10\rangle = \varepsilon_1 \qquad \langle 01|\hat{H}|01\rangle = \varepsilon_2$

$\langle 11|\hat{H}|11\rangle = \varepsilon_1 + \varepsilon_2 + U$

$$\Rightarrow \hat{H} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \varepsilon_1 & t/2 & 0 \\ 0 & t/2 & \varepsilon_2 & 0 \\ 0 & 0 & 0 & \varepsilon_1+\varepsilon_2+U \end{pmatrix} \;\&\; \hat{N} = \begin{pmatrix} 0 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 2 \end{pmatrix}$$

+0,5P ✓ (ε_1), +0,5P ✓ (ε_2), +1P beide t/2 Terme, +0,5P ✓, +0,5P ✓ ($\varepsilon_1+\varepsilon_2+U$), +0,5P: beide N=1 Terme ✓ ✓, +0,5P ✓ (2)

(b)

$$\hat{H} - \mu\hat{N} = \begin{pmatrix} \boxed{0} & 0 & 0 & 0 \\ 0 & \varepsilon_1-\mu & t/2 & 0 \\ 0 & t/2 & \varepsilon_2-\mu & 0 \\ 0 & 0 & 0 & \boxed{\varepsilon_1+\varepsilon_2+U-2\mu} \end{pmatrix}$$

diagonalisieren in 3 Blöcken:

$\underline{\lambda_1 = 0}$, ✓+1P

+1P: char. Polynom

$$\lambda_{2,3}: \begin{vmatrix} \overbrace{\varepsilon_1-\mu}^{\varepsilon_1'}-\lambda & t/2 \\ t/2 & \underbrace{\varepsilon_2-\mu}_{\varepsilon_2'}-\lambda \end{vmatrix} = 0 \Leftrightarrow \lambda^2 - \lambda(\varepsilon_1'+\varepsilon_2') + \varepsilon_1'\cdot\varepsilon_2' - t^2/4 = 0$$

+0,5P: explizite Lösungsformel

$$\Rightarrow \underline{\lambda_{2,3}} = \frac{(\varepsilon_1'+\varepsilon_2') \pm \sqrt{(\varepsilon_1'+\varepsilon_2')^2 - 4\varepsilon_1'\varepsilon_2' + t^2}}{2} = -\mu + \overbrace{\frac{\varepsilon_1+\varepsilon_2}{2}}^{\tilde{\varepsilon}} \pm \frac{\sqrt{(\varepsilon_1'-\varepsilon_2')^2+t^2}}{2}$$ ✓

$$= \tilde{\varepsilon} - \mu \pm \underbrace{\frac{\sqrt{(\varepsilon_1-\varepsilon_2)^2+t^2}}{2}}_{=\gamma/2} = \underline{\tilde{\varepsilon} - \mu \pm \gamma/2}$$

✓+0,5P $\frac{\varepsilon_1'+\varepsilon_2'}{2} = \tilde{\varepsilon} - \mu$

+0,5P: $(\varepsilon_1-\varepsilon_2)^2 = (\varepsilon_0'+\varepsilon_1')^2 - 4\varepsilon_1\varepsilon_0$

$\underline{\lambda_4 = \varepsilon_1 + \varepsilon_2 + U - 2\mu}$ ✓+1P

$$\underline{Z_G} = \mathrm{tr}\left\{ e^{-\beta(\hat{H} - \mu \hat{N})} \right\} = \sum_{I=1}^{4} e^{-\beta \lambda_I}$$ ✓+1P

$$= e^{-\beta 0} + e^{-\beta(\tilde{\varepsilon} - \mu + \gamma/2)} + e^{-\beta(\tilde{\varepsilon} - \mu - \gamma/2)} + e^{-\beta(\overbrace{\varepsilon_1 + \varepsilon_2}^{=2\tilde{\varepsilon}} + U - 2\mu)}$$

$$= 1 + e^{-\beta(\tilde{\varepsilon} - \mu)} \underbrace{\left[e^{-\beta\gamma/2} + e^{+\beta\gamma/2} \right]}_{2\cosh x} + \left(e^{-\beta(\tilde{\varepsilon} - \mu)} \right)^2 \cdot e^{-\beta U}$$

✓+0,5P: $e^{x} + e^{-x} =$

$$= 1 + e^{\xi} 2\cosh(\beta\gamma/2) + e^{2\xi} e^{-\beta U}$$

(c) $\langle N \rangle = -\frac{\partial J}{\partial \mu}$ ✓+1P mit $J = -\frac{1}{\beta} \log Z_G$ ✓+1P

$$= + \frac{1}{\beta} \frac{1}{Z_G} \cdot \frac{\partial Z_G}{\partial \mu} = \cancel{\frac{1}{\beta}} \cdot \frac{1}{Z_G} \left(0 + \cancel{\beta} e^{\xi} 2\cosh(..) + 2\cancel{\beta} e^{2\xi} e^{-\beta U} \right)$$ ✓+1P

$$\frac{\partial e^{\xi}}{\partial \mu} = \beta e^{\xi}$$ ✓+1P

$$= \frac{e^{\xi} 2\cosh(..) + 2e^{2\xi} e^{-\beta U}}{1 + e^{\xi} 2\cosh(..) + e^{2\xi} e^{-\beta U}}$$

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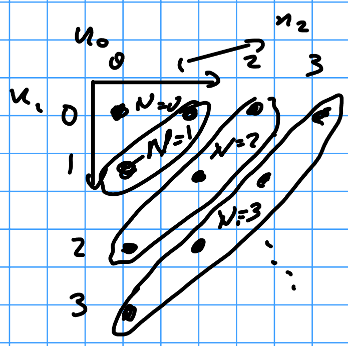
(a) $\underline{Z_G} = \mathrm{tr}\left\{ e^{-\beta(\hat{H}-\mu\hat{N})} \right\}$ $\quad |n_0 n_1 \dots\rangle$



$$= \sum_{N=0}^{\infty} \sum_{\substack{n_0, n_1, \dots = 0 \\ \text{mit } \sum_j n_j = N}} e^{-\beta \sum_j (e_j - \mu) n_j}$$

+1P: tr{..} in Diagonalform

$$= \sum_{n_0=0}^{\infty} \sum_{n_1=0}^{\infty} \cdots \prod_j e^{-\beta(e_j-\mu)n_j}$$

+1P: $\sum_N \underbrace{\sum_{n_1} \dots}_{n_1+\dots=N} \mapsto \sum_{n_1} \sum_{n_2}$

$$= \prod_j \left(\sum_{n_j} e^{-\beta(e_j-\mu)n_j} \right)$$

für Bosonen $n_j \in \{0, \dots\}$

+1P: $\sum \prod \mapsto \prod \sum$

$$= \prod_j \left(\sum_{n_j=0}^{\infty} \underbrace{e^{-\beta(e_j-\mu)n_j}}_{\alpha} \right) \underset{\text{Geom. Reihe}}{=} \underline{\prod_j \left(\frac{1}{1-\alpha} \right)}$$

für $\alpha = e^{-\beta(e_j-\mu)} < 1$

+1P: $\sum_{n=0}^{\infty} \alpha^n = \frac{1}{1-\alpha}$

$\sum_{n_j=0}^{\infty} \alpha^{n_j}$

$\log(1/1-\alpha) = -\log(1-\alpha)$

$$J = -k_B T \log Z_G = +k_B T \sum_j \log\left(1 - e^{-\beta(e_j-\mu)}\right)$$

+0,5P: Ansatz J

$$\langle \hat{N} \rangle = -\frac{\partial J}{\partial \mu} = \cancel{k_B T} \sum_j \frac{e^{-\beta(e_j-\mu)}}{1-e^{-\beta(e_j-\mu)}} \cdot \cancel{\beta} = \sum_j \frac{1}{e^{+\beta(e_j-\mu)}-1}$$

+1P: Ansatz $\langle \hat{N} \rangle$

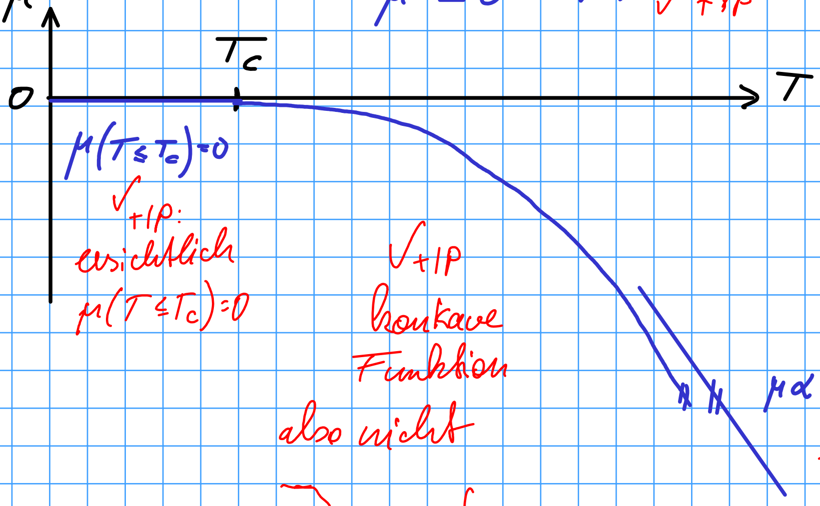
$$= \sum_j \langle \hat{n}_j \rangle$$

+0,5P: Rechnung

$$\Rightarrow \langle \hat{n}_j \rangle = \frac{1}{e^{+\beta(e_j-\mu)}-1}$$

(b)

$\mu \leq 0 \quad \forall T$ ✓ +1P



$\mu(T \leq T_c) = 0$

✓ +1P: ersichtlich $\mu(T \leq T_c) = 0$

✓ +1P konkave Funktion also nicht ~ etc...

$\mu \propto -T$ klassisch

+1P: lineare asymptotic erkennbar