$\boxed{1}$ 35

$$H(\vec{q},\vec{p}) = \sqrt{(mc^2)^2 + (pc)^2}$$

mit $p_n = \sqrt{p_{n,x}^2 + p_{n,y}^2 + p_{n,z}^2}$

a) 7

$$Z_k(\beta,V,N) = \int_\Gamma \frac{d^{3N}q \; d^{3N}p}{h^{3N}\, N!}\, e^{-\beta \sum_{n=1}^{N} \underbrace{\sqrt{(mc^2)^2 + (p_n c)^2}}_{\text{oder } H(\vec{q}_n,\vec{p}_n)}}$$

+1P +1P +1P:(-) ✓+1P: β ✓+1P ✓+1P ✓+1P ✓+1P

$$\left(= \frac{1}{N!}\left[\int \frac{d^3q\; d^3p}{h^3}\, e^{-\beta\sqrt{(mc^2)^2+(pc)^2}}\right]^N \right)$$

b) 13

$$pc \gg mc^2 \implies H(\vec{q},\vec{p}) \approx p_i c + O(p^{-1})$$

✓+1P

$$Z_k(\beta,V,N) = \frac{1}{N!}\frac{V^N}{h^{3N}}\left[\int d^3p \cdot e^{-\beta p_i c}\right]^N$$

+1P: $\int d^3q = V$ ✓

✓+1P: $e^{a+b} = e^a e^b$

✓+1P: $\int d^3p_i\, e^{-\beta p_i c}$ = gruppiert

✓+1P: $\int d^{3N}p = \int d^3p_1 \ldots \int d^3p_N$

Transformation auf Kugelkoordinaten: $d^3p = d\Omega_p\, dp \cdot p^2$ ✓+1P

$$= \frac{1}{N!}\frac{V^N}{h^{3N}}(4\pi)^N\left[\int_0^\infty dp\, p^2 \cdot e^{-\beta p\cdot c}\right]^N$$

✓+1P: $\int d\Omega = 4\pi$

✓+1P ✓+1P $\int_a^b uv'\,dx = uv\Big|_a^b - \int_a^b u'v\,dx$

$$\int_0^\infty dp\cdot p^2 \cdot e^{-\beta p\cdot c} = \underbrace{\frac{1}{(-\beta c)}\, p^2\, e^{-\beta pc}\Big|_0^\infty}_{0} + \frac{2}{\beta c}\int_0^\infty dp \cdot p \cdot e^{-\beta pc} =$$

✓+1P: $\int e^{ax}dx = \frac{1}{a}e^{ax}$

✓ +1P: $(p^2)' = 2p$

0 ✓+1P: für $p=0$

✓+1P: für $p\to\infty$

$$= \underbrace{\frac{-2}{(\beta c)^2}\cdot p \cdot e^{-\beta pc}\Big|_0^\infty}_{0} + \frac{2}{(\beta c)^2}\cdot\int_0^\infty dp \cdot e^{-\beta pc} =$$

$$= \frac{-2}{(\beta c)^3}\cdot e^{-\beta pc}\Big|_0^\infty = \frac{2}{(\beta c)^3} \implies$$

oder Feynman Trick

$$\partial^2_{(\beta c)} e^{-\beta pc} = p^2 e^{-\beta pc}$$

$$\implies \int_0^\infty p^2 e^{-\beta pc}\,dp = \int_0^\infty dp\; \partial^2_{\beta c} e^{-\beta pc}$$

$$= \partial^2_{\beta c}\int_0^\infty e^{-\beta pc}\,dp = \partial^2_{\beta c}\frac{-1}{(-\beta c)}$$

$$\boxed{Z_K(\beta, V, N) = \frac{1}{N!}\left(\frac{8\pi V}{\beta^3 c^3 h^3}\right)^N}$$

$= \partial_{\beta c} \frac{+1}{(-\beta c)^2} = \frac{-2}{(-\beta c)^3} = \frac{2}{(\beta c)^3}$

c) 15

Phasenraumdichte:

$$\rho(q,p) = \frac{1}{Z_K} e^{-\beta \sum_{n=1}^{N} \sqrt{(mc^2)^2 + (p_{n,x}^2 + p_{n,y}^2 + p_{n,z}^2)c^2}}$$

√+1P √+1P √+1P: $e^{-\beta H}$

Für $pc \gg mc^2$:

$$f_1(\vec{p}) = \int d^{3N} q \int d^{3(N-1)} p \; \rho(q,p) =$$

√+1P

$$= \frac{e^{-\beta |\vec{p}| \cdot c}}{\int d^3 p \, e^{-\beta p \cdot c}} = \frac{1}{\frac{4\pi \cdot 2}{(\beta c)^3}} \cdot e^{-\beta |\vec{p}| \cdot c}$$

√+1P: erweitert sodass Z_K^{-1} fällt

√+1P: $\int d^3p \, e^{-\beta pc}$ gefunden

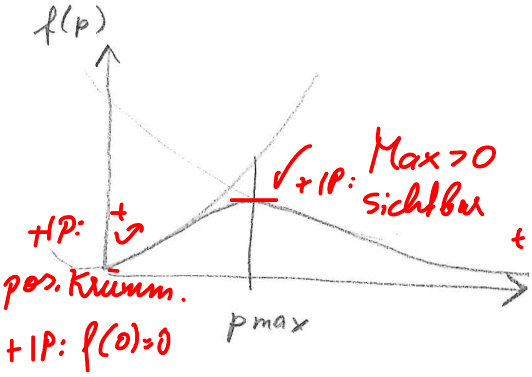
$$f(p)\,dp = f_1(|\vec{p}|)\, d^3 p = f_1(|\vec{p}|) \cdot 4\pi p^2 \, dp \Rightarrow$$

√+1P: Zustandsd. von Beträgen

$$f(p) = \frac{4\pi \cdot p^2}{4\pi} \cdot \frac{(\beta c)^3}{2} \cdot e^{-\beta p \cdot c}$$

$$\boxed{f(p) = \frac{(\beta c)^3}{2} p^2 \cdot e^{-\beta p \cdot c}}$$

√+1P



Maximum: $\frac{d f(p)}{dp} \stackrel{!}{=} 0$ √+1P

$$\frac{df}{dp} = \frac{(\beta c)^3}{2}\left[2p \cdot e^{-\beta pc} + p^2 \cdot (-\beta c) \cdot e^{-\beta pc}\right] \stackrel{!}{=} 0$$

√+1P: Ableitung

$$2\cancel{p} + p^{\cancel{2}}(-\beta c) = 0 \Rightarrow \boxed{p_{max} = \frac{2}{\beta c}}$$

√+1P: $p=0$ kürzen

2 20

a) 2 $[\hat{\rho}, \hat{H}] = 0$ ✓+2P

b) 4 $\hat{\rho} = \sum_k p_k |E_k\rangle\langle E_k|$ mit $\hat{H}|E_k\rangle = E_k |E_k\rangle$

✓+1P ✓+1P ✓+1P ✓+1P

c) 4 $S = -k_B \langle \ln \hat{\rho} \rangle = -k_B \, \mathrm{Tr}\, \hat{\rho} \ln \hat{\rho}$

+1P ✓+1P (+2P falls $S = -k_B \langle \ln\hat{\rho}\rangle$ fehlt)

$$\hat{\rho} \ln \hat{\rho} = \sum_k p_k \ln p_k \, |E_k\rangle\langle E_k|$$

✓+1P: $f(\hat{A}) = \sum_k f(\lambda_k) |A_k\rangle\langle A_k|$

$$\mathrm{Tr}\, \hat{\rho} \ln \hat{\rho} = \sum_k p_k \ln p_k$$

✓+1P (+2P: falls fehlt)

$$S = -k_B \sum_k p_k \ln p_k$$

d) 10

1) Normierung: $\sum_k p_k = 1$ ✓+1P

2) Energiemittelwert: $\sum_k E_k \cdot p_k = E$ ✓+1P

$$\Rightarrow \quad \frac{dS}{dp_i} - C_1 \cdot \underbrace{\frac{d}{dp_i}\left(\sum_k p_k\right)}_{1} - C_2 \underbrace{\frac{d}{dp_i} \sum_k E_k \cdot p_k}_{E_i} \stackrel{!}{=} 0$$

✓+1P ✓+1P ✓+1P ✓+1P ✓+1P

$$-k_B \left[\ln p_i + 1\right] - C_1 - C_2 \cdot E_i = 0$$

+1P (✓+2P: falls fehlen) (✓+2P: falls fehlen)

$$-k_B \ln p_i + \text{const.} - C_2 E_i = 0$$

$$\ln p_i = \text{const.} - \frac{C_2}{k_B} E_i \Rightarrow \boxed{p_i \propto e^{-\frac{C_2}{k_B} \cdot E_i}}$$

✓+1P ✓+1P (+2P: falls fehlt)

$\boxed{3}$ 21

$$Z_G = \prod_k \sum_{n_k} \left(e^{-\beta(\varepsilon_k-\mu)}\right)^{n_k}$$

ε_k ... Eigenenergie des k-ten Ein-Teilchen-Zustandes +1P

a) 5 k .... Index der die Einteilchenzustände nummeriert ✓+1P

n_k .... Besetzungszahl des Einteilchenzustands +1P

Für Fermionen: $n_k = 0, 1$ ✓+1P ✓+1P

b) 4

$$Z_G = \prod_k \left(1 + e^{-\beta(\varepsilon_k-\mu)}\right)$$

✓+1P ✓+1P

$$J = -k_B T \ln Z_G$$

✓+1P

$$= -k_B T \ln \prod_k \left(1+e^{-\beta(\varepsilon_k-\mu)}\right) =$$

$$= -k_B T \sum_k \ln\left(1+e^{-\beta(\varepsilon_k-\mu)}\right)$$

✓+1P: $\ln \prod = \sum \ln$ (+1P, wenn in (c) richtig verwendet)

c) 4

$$\langle \hat N \rangle = -\frac{\partial J}{\partial \mu}$$ ✓+1P

$$= + \cancel{k_B T} \sum_k \underbrace{\frac{1}{1+e^{-\beta(\varepsilon_k-\mu)}}}_{+1P:\ (\ln x)' = \frac{1}{x}} \cdot \underbrace{e^{-\beta(\varepsilon_k-\mu)} \cdot \beta}_{✓+1P:\ \text{innere Abl.}} =$$

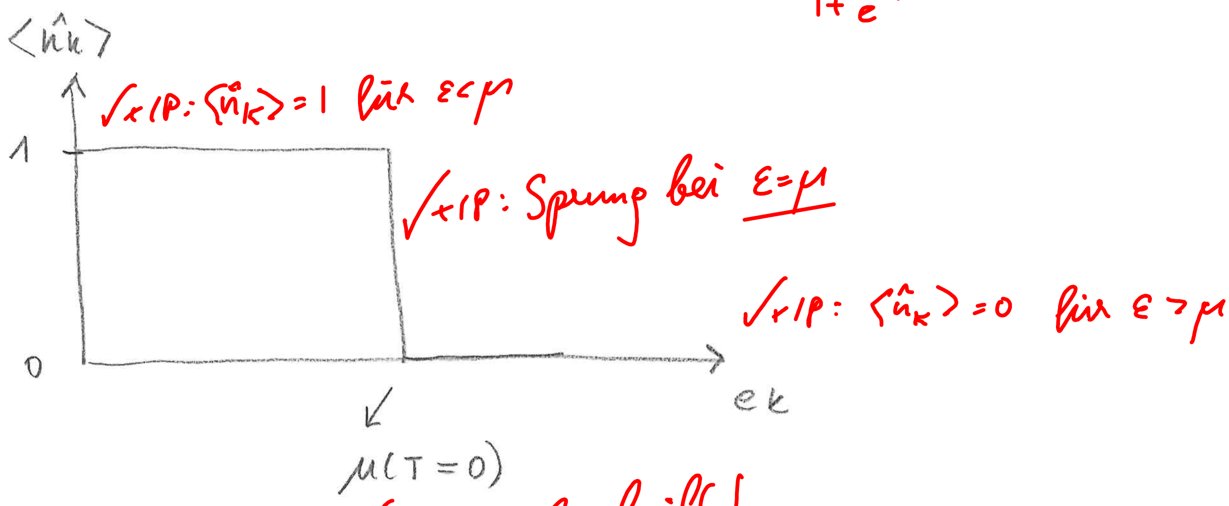
✓+1P:

$$= \sum_k \frac{1}{e^{\beta(\varepsilon_k-\mu)}+1} = \sum_k \langle \hat n_k \rangle$$

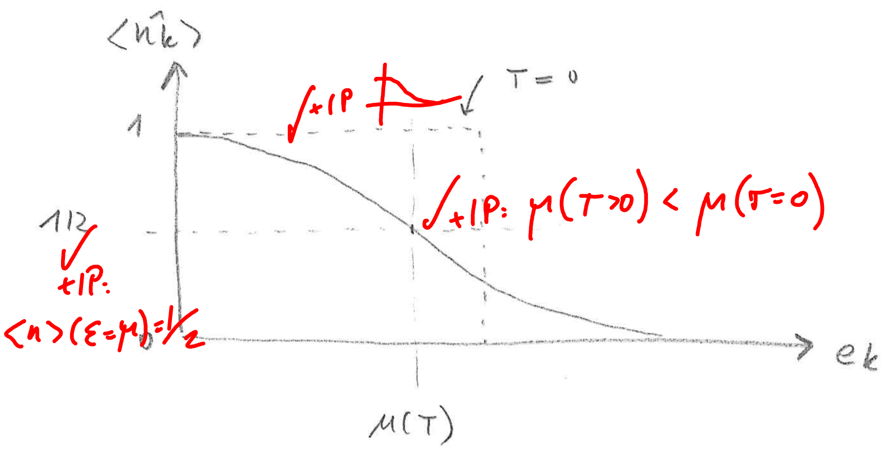
d) 8

$$\langle \hat{n}_k \rangle = \frac{1}{e^{\beta(\epsilon_k - \mu)} + 1}$$

✓ +1P: $\langle \hat{n}_k \rangle$ identifiziert, auch $\frac{e^{-\beta \cdot \cdot}}{1+e^{-\beta \cdot \cdot}}$ ok



Für $T > 0$



4   24

a) (6)

$$Z_G = \sum_{n_1=0}^{1} \sum_{n_2=0}^{1} \underbrace{e^{-\beta(\varepsilon_1-\mu)\cdot n_1} \, e^{-\beta(\varepsilon_2-\mu)\cdot n_2}}_{+1P:\ \left(e^{-\beta(\varepsilon..)}\right)^{n_1}\left(\,--\,\right)^{n_2}} =$$

+1P ✓   ✓+1P: $e^{-\beta(\varepsilon-\mu)}$ ✓+1P

✓+1P: $\prod_k \sum_{n_k}$ +1P: 4 Terme $00, 10, 01, 11 = n_1 n_2$

$$= 1 + \underbrace{e^{-\beta\varepsilon_1}}_{g_1} \cdot z + \underbrace{e^{-\beta\varepsilon_2}}_{g_2} z + \underbrace{e^{-\beta\varepsilon_1} z \cdot e^{-\beta\varepsilon_2} \cdot z}_{g_1 \cdot g_2 \cdot z^2}$$

b) (6)

(oder: +4P mit $\langle \hat{n}_k \rangle = \frac{1}{e^{+\beta(\varepsilon_k-\mu)}+1}$, $\langle \hat{N} \rangle = \sum_k \langle \hat{n}_k \rangle$)

$$J = -k_B T \ln\left[1 + z(g_1+g_2) + z^2 g_1 \cdot g_2\right]$$

✓+1P

$$\langle \hat{N} \rangle = -\frac{\partial J}{\partial \mu} = \cancel{k_B T} \cdot \frac{1}{1 + z(g_1+g_2) + z^2 g_1 \cdot g_2} \times$$

✓+1P   ✓   ✓+1P: $(\ln x)' = \frac{1}{x}$

$$\times \overbrace{\cancel{\beta} z(g_1+g_2) + 2\cancel{\beta} \cdot z^2 g_1 \cdot g_2}^{+1P:\ \text{innere Abl.}}$$

✓+1P: $\beta\beta^{-1}$

$$\underbrace{\langle \hat{N} \rangle = \frac{z(g_1+g_2) + 2z^2 g_1 g_2}{1 + z(g_1+g_2) + z^2 g_1 \cdot g_2} \stackrel{!}{=} \frac{1}{2}}_{+1P}$$

$$z(g_1+g_2) + 2z^2 g_1 \cdot g_2 = \frac{1}{2} + \frac{1}{2} z(g_1+g_2) + \frac{1}{2} z^2 g_1 g_2 \Rightarrow$$

$$\boxed{z(g_1+g_2) + 3z^2 g_1 \cdot g_2 - 1 = 0}$$

c) 12

$$3z^2 g_1 \cdot g_2 + z(g_1+g_2) - 1 = 0$$

$$z^2 + z\frac{(g_1+g_2)}{3g_1 \cdot g_2} - \frac{1}{3g_1 \cdot g_2} = 0$$

$$z_{1,2} = -\frac{g_1+g_2}{6g_1g_2} \pm \sqrt{\frac{(g_1+g_2)^2}{36\,g_1^2 g_2^2} + \frac{1}{3g_1g_2}}$$

✓+1P: $z_{1,2}$ Formel

Da $z = e^{\mu\beta} \geq 0$ ist nur das obere Vorzeichen möglich:

✓+1P ✓+1P

+1P: $e^{-\beta\epsilon_k} \to 1$

$$z(\beta \to 0) = -\frac{(1+1)}{6} + \sqrt{\frac{(1+1)^2}{36} + \frac{1}{3}}$$

$$= -\frac{1}{3} + \sqrt{\frac{1}{9} + \frac{1}{3}} = -\frac{1}{3} + \frac{2}{3} = \frac{1}{3}$$

✓+1P: Rechnung

$\Rightarrow$

$$z(\beta) = \frac{1}{3} + O(\beta)$$

$\mu(\beta) = \dfrac{\ln z(\beta)}{\beta}$ ✓+1P: $\mu(z)$ gelöst

✓+1P: Produktform

$$\ln z(\beta) = \ln\left[\frac{1}{3} + O(\beta)\right] = \ln\left[\frac{1}{3}\left(1 + O(\beta)\right)\right] =$$

✓+1P: $3O(\beta) = O(\beta)$

$$= \ln\frac{1}{3} + \underbrace{\ln(1+O(\beta))}$$

✓+1P: $\ln(ab) = \ln a + \ln b$

$\ln(1+x) \approx x + O(x^2)$ für kleine x $\Rightarrow$

✓+1P

$$\ln z(\beta) = -\ln 3 + O(\beta)$$

$$\Rightarrow \boxed{\mu(\beta) = -\underbrace{\frac{\ln 3}{\beta}}_{✓+1P} + O(1)}$$

✓ +1P: $\dfrac{O(\beta)}{\beta} = O(1)$